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Generalized symmetric functions and invariants of matrices[1]

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Abstract We generalize the classical isomorphism between symmetric functions and invariants of a matrix. In particular we show that the invariants over several matrices are given by a the abelianization of the symmetric tensors over the free associative algebra. The main result is proved by founding a characteristic free presentation of the algebra of symmetric tensors over a free algebra.

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1 Introduction

Let $\mathbb{K}$ be an infinite field and let $\mathbb{K}[y_1,\dots,y_n]^{S_n}$ be the ring of *symmetric polynomials* in n variables. The general linear group $\mathrm{GL}(n,\mathbb{K})$ acts by conjugation on the full ring $\mathrm{Mat}(n,\mathbb{K})$ of $n\times n$ matrices over $\mathbb{K}$. Denote by $\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})]^{\mathrm{GL}(n,\mathbb{K})}$ the ring of polynomial invariants for this actions. It is well known that

$$\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})]^{\mathrm{GL}(n,\mathbb{K})} \cong \mathbb{K}[y_1,\dots,y_n]^{S_n}. \qquad (1)$$

Let M be a vector space over $\mathbb{K}$. Consider the tensor product $M^{\otimes n}$, the symmetric group acts on $M^{\otimes n}$ as a group of linear automorphisms and we denote by $\mathrm{TS}^n(M) = (M^{\otimes n})^{S_n}$ the subspace of the invariants for this action. The elements of $\mathrm{TS}^n(M)$ are called symmetric tensors of order n. If M is a $\mathbb{K}-$algebra then $\mathrm{TS}^n(M)$ is a $\mathbb{K}-$subalgebra of M.

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Let $F = \mathbb{K}\{x_1,\dots,x_m\}$ be a free associative non commutative algebra on m variables. The isomorphism (1) can be written as

$$\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})]^{\mathrm{GL}(n,\mathbb{K})} \cong \mathrm{TS}^n(\mathbb{K}[x]) = \mathrm{TS}^n(\mathbb{K}\{x\}) \tag{2}$$

and this observation leads us to study the following objects. For a $\mathbb{K}-$algebra A we write $A^{ab} = A/[A,A]$ for the abelianization of A, where $[A,A]$ denotes the ideal generated by the commutators. Consider

(i) $\mathrm{TS}^n(F)$ and
(ii) $\mathrm{TS}^n(F)^{ab}$ the abelianization of $\mathrm{TS}^n(F)$

If $m = 1$ then F is commutative and $\mathrm{TS}^n(F) = \mathrm{TS}^n(F)^{ab}$.
We prove the following generalization of the isomorphisms (1) and (2).

Theorem 1 *Let $\mathbb{K}$ be an infinite field or the ring of integers and let the general linear group* $\mathrm{GL}(n,\mathbb{K})$ *acts by simultaneous conjugation on m copies of* $\mathrm{Mat}(n,\mathbb{K})$. *Denote by* $\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})^m]^{\mathrm{GL}(n,\mathbb{K})}$ *the ring of the invariants for this action. Then*

$$\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})^m]^{\mathrm{GL}(n,\mathbb{K})} \cong \mathrm{TS}^n(\mathbb{K}\{x_1,\dots,x_m\})^{ab}$$

Remark 1 When $\mathbb{K} = \mathbb{Z}$ note that we are talking about the invariants for the action of the general linear group scheme over $\mathbb{Z}$.

Remark 2 Let $\mathbb{K}$ be an infinite field and let $Z^m_{n,red}$ be the variety of $m-$tuples of pairwise commuting $n\times n$ matrices. In [15] we proved that there is an isomorphism

$$\mathrm{TS}^n(\mathbb{K}[x_1,\dots,x_m]) \cong \mathbb{K}[Z^m_{n,red}]^{\mathrm{GL}(n,\mathbb{K})}$$

Moreover if $\mathrm{char}\,\mathbb{K} = 0$ then we showed that the above isomorphism extends to the corresponding affine schemes i.e.

$$\mathrm{TS}^n(\mathbb{K}[x_1,\dots,x_m]) \cong \mathbb{K}[Z^m_n]^{\mathrm{GL}(n,\mathbb{K})}$$

where Z^m_n is the affine scheme of $m-$tuples of pairwise commuting $n\times n$ matrices. The present article and [15] present extensions of the characteristic free presentation of the ring of mutlisymmetric functions that we presented in [14].

2 Symmetric functions

Let $\mathbb{K}$ be an arbitrary commutative ring and let $y_1,...,y_n$ be independent variables. The symmetric group S_n acts on the polynomial ring $\mathbb{K}[y_1,...,y_n]$ by permuting the y's , and we shall write

$$\Lambda_n = \mathbb{K}[y_1,\dots,y_n]^{S_n}$$

for the subring of symmetric polynomials in $y_1,\dots,y_n$. Let t be another variable. The ring Λ_n is freely generated as a $\mathbb{K}-$algebra by the elementary symmetric functions $e_1,\dots,e_n$ given by the following equality in $\Lambda_n[t]$

$$\sum_{k=0}^{n} t^k e_k = \prod_{i=1}^{n}(1+ty_i) \tag{3}$$

where $e_0 = 1$ (see [7]). Furthermore one has

$$e_k(y_1,\dots,y_n) = \sum_{i_1<i_2<\cdots<i_k\le n} y_{i_1}y_{i_2}\cdots y_{i_k} \tag{4}$$

The action of S_n on $\mathbb{K}[y_1,y_2,\dots,y_n]$ preserves the usual degree. We denote by Λ_n^k the $\mathbb{K}$-submodule of invariants of degree k.

Let $q_n : \mathbb{K}[y_1,y_2,\dots,y_n] \to \mathbb{K}[y_1,y_2,\dots,y_{n-1}]$ be given by mapping y_i to y_i for $i = 1,\dots,n-1$ and y_n to 0. One has $q_n(\Lambda_n^k) = \Lambda_{n-1}^k$ and it is easy to see that $\Lambda_n^k \cong \Lambda_k^k$ for all $n \ge k$. Denote by Λ^k the limit of the inverse system obtained in this way.

Definition 1 The ring $\Lambda = \bigoplus_{k\ge 0} \Lambda^k$ is called the ring of symmetric functions (over $\mathbb{K}$).

It can be showed (see [7]) that Λ is a polynomial ring freely generated by the (limit of the) e_k's, which have generating function

$$\sum_{k=0}^{\infty} t^k e_k = \prod_{i=1}^{\infty}(1+ty_i). \tag{5}$$

Furthermore the kernel of the natural map $\pi_n : \Lambda \to \Lambda_n$ is the ideal generated by the e_{n+k}, where $k \ge 1$.

We have another distinguished kind of functions in Λ_n beside the elementary symmetric ones: the *power sums*. For $r \in \mathbb{N}$ the r-th power sum is

$$p_r = \sum_{i\ge 1} y_i^r \tag{6}$$

Let $g \in \Lambda_n$, set $g \cdot p_r = g(y_1^r, y_2^r, \dots, y_n^r)$ for the plethysm of g and p_r (see Section I.8 of [7]). The function $g \cdot p_r$ is again symmetric. Since the e_i freely generate Λ_n we have that $g \cdot p_r$ can be expressed as a polynomial in the e_i and we denote it by

$$P_{h,k} = e_h \cdot p_k \tag{7}$$

The monomials form a $\mathbb{K}-$basis of $\mathbb{K}[y_1,\dots,y_n]$ permuted by S_n. Hence the sums of monomials over the orbits form a $\mathbb{K}-$basis of the ring Λ_n and their limits form a basis of Λ. Let $y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_n^{\lambda_n}$ be a monomial, after a suitable permutation we can suppose $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0$. We set m_λ for the orbit sum corresponding to such $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{N}^n$ then

$$\mathscr{P}_n = \{m_\lambda : \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n \ge 0, \lambda_i \in \mathbb{N}\} \tag{8}$$

and

$$\mathscr{P}_{n,k} = \{m_\lambda : \sum_i \lambda_i = k\} \tag{9}$$

are $\mathbb{K}-$bases of Λ_n and Λ_n^k respectively. As before the limits of the m_λ form a basis of Λ and Λ^k and $\ker \pi_n$ has basis $\{m_\lambda : \lambda_{n+1} > 0\}$

3 Symmetric Tensors on Free Algebras

We give here a generalization of Λ and Λ_n. Our exposition will be based on the one given in the previous section.

Definition 2 Let M be a $\mathbb{K}-$module and consider the tensor power $M^{\otimes n}$. The symmetric group S_n acts on $M^{\otimes n}$ by permuting the factors and we denote by $\mathrm{TS}^n_{\mathbb{K}}(M)$ or simply by $\mathrm{TS}^n(M)$ the $\mathbb{K}-$submodule of $M^{\otimes n}$ of the invariants for this action. The elements of $\mathrm{TS}^n(M)$ are called symmetric tensors of degree n over M.

Remark 3 If M is a $\mathbb{K}-$algebra then S_n acts on $M^{\otimes n}$ as a group of $\mathbb{K}-$algebra automorphisms. Hence $\mathrm{TS}^n(M)$ is a $\mathbb{K}-$subalgebra of $M^{\otimes n}$.

Remark 4 The map $f : \mathbb{K}[x_1,\dots,x_n] \to \mathbb{K}[x]^{\otimes n}$ given by $f(x_i) = 1^{\otimes i-1} \otimes x \otimes 1^{n-i}$ for $i = 1,\dots,n$ is an S_n-equivariant isomorphism such that $\Lambda_n \cong \mathrm{TS}^n(\mathbb{K}[x])$.

Let now $F = \mathbb{K}\{x_1,\dots,x_m\}$ be the free associative non commutative $\mathbb{K}-$algebra on m generators. Let $k \in \mathbb{N}$, we denote by $\mathbf{f}$ the sequence $(f_1 \dots, f_k)$ of elements of F and by α the element $(\alpha_1,\dots,\alpha_k) \in \mathbb{N}^k$, with $|\alpha| = \sum \alpha_j \leq n$. Let $t_1,\dots,t_k$ be commuting independent variables, we set as usual $t^\alpha = \prod_i t_i^{\alpha_i}$. We define elements $e^n_\alpha(\mathbf{f}) \in \mathrm{TS}^n(F)$ by

$$\sum_{|\alpha|\leq n} t^\alpha \otimes e^n_\alpha(\mathbf{f}) = (1+\sum_h t_h \otimes f_h)^{\otimes n} \tag{10}$$

where the equality is computed in $\mathbb{K}[t_1,\dots,t_k] \otimes \mathrm{TS}^n(F)$.

Example 1 Let $f, g \in F$ then

$$\begin{aligned} e^3_{(0,0,0)}(f,g) &= 1\otimes 1\otimes 1 \\ e^3_{(2,1)}(f,g) &= f\otimes f\otimes g + f\otimes g\otimes f + g\otimes f\otimes f \\ e^4_{(2,1)}(f,g) &= f\otimes f\otimes g\otimes 1 + f\otimes g\otimes f\otimes 1 + g\otimes f\otimes f\otimes 1 \\ &+ f\otimes f\otimes 1\otimes g + f\otimes g\otimes 1\otimes f + g\otimes f\otimes 1\otimes f \\ &+ f\otimes 1\otimes f\otimes g + f\otimes 1\otimes g\otimes f + g\otimes 1\otimes f\otimes f \\ &+ 1\otimes f\otimes f\otimes g + 1\otimes f\otimes g\otimes f + 1\otimes g\otimes f\otimes f \end{aligned}$$

Lemma 1 *The element $e^n_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k)$ is the orbit sum under the considered action of S_n of*

$$f_1^{\otimes\alpha_1} \otimes f_2^{\otimes\alpha_2} \otimes \cdots \otimes f_k^{\otimes\alpha_k} \otimes 1^{\otimes(n-\sum_i \alpha_i)}$$

Proof Let E be the set of mappings $\phi : \{1,\dots,n\} \to \{1,\dots,k+1\}$. We define a mapping $\phi \mapsto \phi^*$ of E into $\mathbb{N}^{k+1}$ by putting $\phi^*(i)$ equal to the cardinality of $\phi^{-1}(i)$. For two elements ϕ_1,ϕ_2 of E, to satisfy $\phi_1^* = \phi_2^*$ it is necessary and sufficient that there should exist $\sigma \in S_n$ such that $\phi_2 = \phi_1 \circ \sigma$. Set $f_{k+1} = 1$ and $E(\alpha) = \{\phi \in E : \phi^* = (\alpha_1,\dots,\alpha_k, n-\sum_i \alpha_i)\}$, then we have

$$e_\alpha(\mathbf{f}) = \sum_{\phi\in E(\alpha)} f_{\phi(1)} \otimes f_{\phi(2)} \otimes \cdots \otimes f_{\phi(n)}$$

and the lemma is proved. □

Remark 5 The mapping ϕ^* is the same as the *content* in [6].

Definition 3 Let $f \in F$. We denote by $e_i^n(f)$ the element $e^n_{(i,n-i)}(f,1)$ of $\mathrm{TS}^n(F)$ which is the orbit sum of $f^{\otimes i} \otimes 1^{\otimes n-i}$.

Example 2 Let $f \in F$ then

$$\begin{aligned} e_1^3(f) &= f\otimes 1\otimes 1 + 1\otimes f\otimes 1 + 1\otimes 1\otimes f \\ e_2^3(f) &= f\otimes f\otimes 1 + f\otimes 1\otimes f + 1\otimes f\otimes f \\ e_3^3(f) &= f\otimes f\otimes f \end{aligned}$$

Remark 6 Let $f \in F$. The evaluation $\mathbb{K}[x] \to \mathbb{K}[f]$ induces an S_n-equivariant homomorphism $\rho_f : \mathbb{K}[y_1,\dots,y_n] \cong \mathbb{K}[x]^{\otimes n} \to F^{\otimes n}$ such that

$$\rho_f(y_h) = 1^{\otimes(h-1)} \otimes f \otimes 1^{\otimes(n-h)}$$

We then have that $\rho_f(e_i) = e_i^n(f)$.

Definition 4 Let $\mathfrak{M}$ denote the set of monomials in F. There is a natural degree "d"on F given by $\mathrm{d}(x_i) = 1$ for all $i = 1,\dots,m$ and $\mathrm{d}(0) = 1$. We denote by $\mathfrak{M}^+$ the set of monomials of positive degree. Thus $\mathfrak{M} = \mathfrak{M}^+ \cup \{1\}$.

It is clear that $\mathfrak{M}$ is a $\mathbb{K}-$basis of F so that $\mathfrak{M}_n = \{\upsilon_1 \otimes \upsilon_2 \otimes \cdots \otimes \upsilon_n : \upsilon_j \in \mathfrak{M}\}$ is a $\mathbb{K}-$basis of $F^{\otimes n}$ permuted by S_n. Thus, the sums of the elements of $\mathfrak{M}_n$ over their orbits form a $\mathbb{K}-$basis of $\mathrm{TS}^n(F)$.

Let $\alpha \in \mathbb{N}^{(\mathfrak{M}^+)}$, then there exist $k \in \mathbb{N}$ and $\upsilon_1,\dots,\upsilon_k \in \mathfrak{M}^+$ such that $\alpha(\upsilon_i) = \alpha_i \neq 0$ for $i = 1,\dots,k$ and $\alpha(\mathfrak{M}) = 0$ when $\upsilon \neq \upsilon_1,\dots,\upsilon_k$. We write

$$e^n_\alpha = e^n_{(\alpha_1,\dots,\alpha_k)}(\upsilon_1,\dots,\upsilon_k) \tag{11}$$

Proposition 1 *The set*

$$\mathscr{B}_n = \{e^n_\alpha \ : \ |\alpha| \le n\}$$

is a $\mathbb{K}-$*basis of* $\mathrm{TS}^n(F)$.

Proof By Lemma 1 the e^n_α are a complete system of representatives (for the action of S_n) of the orbit sums of the elements of $\mathfrak{M}_n$. □

4 Generators

First of all we compute the product of elements of $\mathscr{B}_n$

Proposition 2 (Product Formula) *Let* $h,k \in \mathbb{N}$, $\alpha \in \mathbb{N}^h$, $\beta \in \mathbb{N}^k$ *be such that* $|\alpha|, |\beta| \le n$. *Let* $r_1,\dots,r_h,s_1,\dots,s_k \in F$. *Set again*

$$e^n_\alpha(\mathbf{r}) = e^n_{(\alpha_1,\dots,\alpha_h)}(r_1,\dots,r_h) \quad \textit{and} \quad e^n_\beta(\mathbf{s}) = e^n_{(\beta_1,\dots,\beta_k)}(s_1,\dots,s_k)$$

then

$$e^n_\alpha(\mathbf{r})e^n_\beta(\mathbf{s}) = \sum_\gamma e^n_\gamma(\mathbf{r},\mathbf{s},\mathbf{rs})$$

where

$$\mathbf{rs} = (r_1s_1, r_1s_2, \ldots, r_1s_k, r_2s_1, \ldots, r_2s_k, \ldots, r_hs_k)$$

$$\gamma = (\gamma_{10}, \gamma_{20}, \ldots, \gamma_{h0}, \gamma_{01}, \ldots, \gamma_{0k}, \gamma_{11}, \ldots, \gamma_{1k}, \ldots, \gamma_{h1}, \ldots, \gamma_{hk})$$

are such that

$$\begin{cases} \gamma_{ij} \in \mathbb{N} \\ \sum_{i,j} \gamma_{ij} \le n \\ \sum_{j=0}^{k} \gamma_{ij} = \alpha_i \ \textit{for} \ i = 1, \ldots, h \\ \sum_{i=0}^{h} \gamma_{ij} = \beta_j \ \textit{for} \ j = 1, \ldots, k. \end{cases} \tag{12}$$

Proof Let t_1, t_2 be two commuting independent variables and let $a, b \in F$. We have

$$(1 + t_1 \otimes a)^{\otimes n}(1 + t_2 \otimes b)^{\otimes n} = (1 + t_1 \otimes a + t_2 \otimes b + t_1t_2 \otimes ab)^{\otimes n} \tag{13}$$

hence

$$\begin{aligned} (1 + \sum_{i=1}^{n} t_1^i \otimes e_i^n(a))(1 + \sum_{j=1}^{n} t_2^j \otimes e_j^n(b)) \\ &= 1 + \sum_{i,j} t_1^i t_2^j \otimes e_i^n(a) e_j^n(b) \\ &= 1 + \sum_{l_1, l_2, l_{12}} t_1^{l_1 + l_{12}} t_2^{l_2 + l_{12}} \otimes e^n_{(l_1, l_2, l_{12})}(a, b, ab) \end{aligned}$$

The desired equation then easily follows. □

Remark 7 The product formula could be easier visualized observing that we are summing over those matrices of positive integers

$$\overline{\gamma} = \begin{pmatrix} 0 & \gamma_{01} & \cdots & \gamma_{0k} \\ \gamma_{10} & \gamma_{11} & \cdots & \gamma_{1k} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_{h0} & \gamma_{h1} & \cdots & \gamma_{hk} \end{pmatrix}$$

having the last h rows and the last k columns having sum $\alpha_1, \ldots, \alpha_h$ and $\beta_1, \ldots, \beta_k$ respectively.

Remark 8 The above Product Formula can be derived from the one found by N.Roby in the context of divided powers (see [12]). It has also been derived by D.Ziplies in his paper on the divided powers algebra $\widehat{\Gamma}$ (see[16]).

Corollary 1 *Let $k \in \mathbb{N}$, $a_1, \ldots, a_k \in F$, $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ with $|\alpha| \le n$. Then $e^n_{(\alpha_1, \ldots, \alpha_k)}(a_1, \ldots, a_k)$ belongs to the subalgebra of $\mathrm{TS}^n(F)$ generated by the $e_i^n(\upsilon)$, where $i = 1, \ldots, n$ and υ is a monomial in the $a_1, \ldots, a_k$.*

Proof We prove the claim by induction on $|\alpha|$ assuming that $\alpha_i > 0$ for all i (note that $1 \leq k \leq \sum_j \alpha_j$). Since n is fixed we suppress the superscript n for all the proof.

If $\sum_j \alpha_j = 1$ then $k = 1$ and $e_{(\alpha_1,\dots,\alpha_k)}(a_1,\dots,a_k) = e_1(a_1)$. Suppose the claim true for all $e_{(\beta_1,\dots,\beta_h)}(b_1,\dots,b_h)$ with $b_1,\dots,b_h \in F$ and $|\beta| < |\alpha|$.

Let $k, a_1, \dots, a_k, \alpha$ be as in the statement, then we have by the Product Formula

$$e_{\alpha_1}(a_1)e_{(\alpha_2,\dots,\alpha_k)}(a_2,\dots,a_k) =$$

$$= e_{(\alpha_1,\dots,\alpha_k)}(a_1,\dots,a_k) + \sum e_\gamma(a_1,\dots,a_k,a_1a_2,\dots,a_1a_k),$$

where

$$\gamma = (\gamma_{10}, \gamma_{01}, \dots, \gamma_{0h}, \gamma_{11}, \gamma_{12}, \dots, \gamma_{1h})$$

with $h = k-1$, $\sum_{j=0}^h \gamma_{1j} = \alpha_1$ with $\sum_{j=1}^h \gamma_{1j} > 0$, and $\gamma_{0j} + \gamma_{1j} = \alpha_j$ for $j = 1,\dots,h$. Thus

$$\gamma_{10} + \gamma_{01} + \cdots + \gamma_{0h} + \gamma_{11} + \cdots + \gamma_{1h} = \sum_j \alpha_j - \sum_{j=1}^h \gamma_{1j} < \sum_j \alpha_j.$$

Hence

$$e_{(\alpha_1,\dots,\alpha_k)}(a_1,\dots,a_k) =$$

$$e_{\alpha_1}(a_1)e_{(\alpha_2,\dots,\alpha_k)}(a_2,\dots,a_k) - \sum e_\gamma(a_1,\dots,a_k,a_1a_2,a_1a_3,\dots,a_1a_k),$$

where $|\gamma| = \sum_{r,s} \gamma_{rs} < |\alpha|$. So the claim follows by induction hypothesis. □

Corollary 2 *The algebra of symmetric tensors* $\mathrm{TS}^n(F)$ *of order n is generated by the* $e_i^n(\upsilon)$ *where* $1 \leq i \leq n$ *and* $\upsilon \in \mathfrak{M}^+$.

Proof It follows from Corollary 1 applied to the elements of the basis $\mathscr{B}_n$. □

Remark 9 The above corollaries can also be proved using Corollary (4.1) and (4.5) in [16].

Lemma 2 *For all* $f \in F$, *and* $k, h \in \mathbb{N}$, $e_h^n(f^k)$ *belongs to the subalgebra of* $\mathrm{TS}^n(F)$ *generated by the* $e_j^n(f)$.

Proof From Remark 6 and (7) it follows that

$$e_h^n(f^k) = \rho_f(e_h \cdot p_k) = \rho_f(P_{h,k}(e_1,\dots,e_n)) = P_{h,k}(e_1^n(f),\dots,e_n^n(f))$$

and the result is proved. □

Definition 5 A monomial $\upsilon \in \mathfrak{M}^+$ is called *primitive* if it is not the proper power of another one.

Example 3 $x_1x_2x_1x_2$ is not primitive while $x_1x_2x_1x_1$ is primitive.

We have then the following refinement of Corollary 2.

Theorem 2 (Generators) *The algebra* $\mathrm{TS}^n(F)$ *is generated by* $e_i^n(\upsilon)$ *with* $1 \leq i \leq n$ *and* υ *primitive.*

Proof It follows from Corollary 2 and Lemma 2. □

4.1 Abelianization

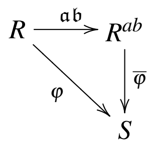
Recall from the introduction that given a $\mathbb{K}-$algebra R we denote by $[R,R]$ the two-sided ideal of R generated by the commutators $[a,b] = ab - ba$ with $a,b \in R$. We write

$$R^{ab} = R/[R,R]$$

and call it the abelianization of R. The abelianization of R is commutative. The surjective homomorphism

$$\mathfrak{ab} : R \longrightarrow R/[R,R]$$

is such that for all commutative $\mathbb{K}-$algebra S and any $\mathbb{K}-$algebra homomorphism $\varphi : R \to S$ there is a unique homomorphism of (commutative) $\mathbb{K}-$algebras $\overline{\varphi} : R^{ab} \to S$ such that the following diagram commutes



Definition 6 Consider $\mathfrak{M}/\sim$ the set of the equivalence classes of monomials $\upsilon \in \mathfrak{M}^+$ where $\upsilon \sim \upsilon'$ if and only if there is a cyclic permutation σ such that $\sigma(\upsilon) = \upsilon'$. We set Ψ to denote the set of equivalence classes in $\mathfrak{M}^+/\sim$ made of primitive monomials

Definition 7 We write $\mathfrak{e}^n_\alpha$ or $\mathfrak{e}^n_i(\upsilon)$ for $\mathfrak{ab}(e^n_\alpha)$ or $\mathfrak{ab}(e^n_i(\upsilon))$ respectively.

Theorem 3 *The algebra* $\mathrm{TS}^n(F)^{ab}$ *is generated by* $\mathfrak{e}^n_i(\upsilon)$ *where* $1 \le i \le n$ *and* υ *varying in a complete set of representatives of* Ψ.

Proof Using (13) it easy to see that

$$\mathfrak{e}^n_i(rs) = \mathfrak{e}^n_i(sr) \tag{14}$$

for all $1 \le i \le n$ and $r,s \in F$. The result then follows from Theorem 2 and the surjectivity of $\mathfrak{ab}$. □

4.2 Good Characteristics

In the ring Λ of symmetric functions it holds the following well known **Newton's Formula**

$$(-1)^k p_{k+1} + \sum_{i=1}^{k} (-1)^i p_i e_{k+1-i} = (k+1)e_{k+1} \tag{15}$$

for all $k > 0$. It is clear that these equalities hold also in Λ_n with $e_i = 0$ for $i > n$.

Proposition 3 *If* $n!$ *is invertible in* $\mathbb{K}$ *then* $\mathrm{TS}^n(F)$ *is generated by* $e^n_1(\upsilon)$ *where* $\upsilon \in \mathfrak{M}^+$. *In this case* $\mathrm{TS}^n(F)^{ab}$ *is generated by* $e^n_1(\upsilon)$ *where* $\upsilon \in \mathfrak{M}^+/\sim$.

Proof Using Newton's formulas one can show that $p_1, p_2, \ldots, p_n$ is a generating set for Λ_n hence $e^n_i(\upsilon)$ belongs to the subring generated by the $e^n_1(\upsilon^k)$. This fact together with Theorem 2 give the desired result. The same argument together with Theorem 3 give the result relative to $\mathrm{TS}^n(F)^{ab}$. □

5 Relations: the first syzygy

We have a system of generators. We now look for relations between them: the first syzygy.

Definition 8 We define an S_n-invariant degree ∂ on $F^{\otimes n}$ by

$$\partial(1^{\otimes i}\otimes \upsilon \otimes 1^{n-i-1}) = \mathrm{d}(\upsilon)$$

for all i and $\upsilon \in \mathfrak{M}$, where d is given in Definition 4. We denote by $\mathrm{TS}^n(F)_d$ (resp. $F_d^{\otimes n}$) the linear span of the elements of degree $d \in \mathbb{N}$.

Remark 10 Let $f_1,\dots,f_k$ be homogeneous of degrees $\mathrm{d}(f_1),\dots,\mathrm{d}(f_k)$. Then $e^n_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k)$ is homogeneous of degree

$$\partial(e^n_{(\alpha_1,\dots,\alpha_k)}(f_1,\dots,f_k)) = \alpha_1\mathrm{d}(f_1)+\cdots\alpha_k\mathrm{d}(f_k).$$

Remark 11 Since ∂ is S_n-invariant we have

$$\mathrm{TS}^n(F) = \bigoplus_{d\in\mathbb{N}} \mathrm{TS}^n(F)_d$$

and $\mathrm{TS}^n(F)$ is a graded ring with respect to ∂.

Proposition 4 *The set*

$$\mathscr{B}_{n,d} = \{e^n_\alpha \,:\, |\alpha| \leq n \text{ and } \partial(e_\alpha) = d\}$$

is a $\mathbb{K}-$*basis of* $\mathrm{TS}^n(F)_d$ *for all* $d \in \mathbb{N}$.

Proof Observe that $\partial(e^n_\alpha) = \sum_{\upsilon\in\mathfrak{M}^+}\alpha_\upsilon \mathrm{d}(\upsilon)$ and apply Proposition 1. □

Corollary 3 *For* $d \in \mathbb{N}$ *we have*

$$\mathrm{rank}_{\mathbb{K}}\mathrm{TS}^n(F)_d = \mathrm{rank}_{\mathbb{K}}\mathrm{TS}^d(F)_d$$

for all $n \geq d$.

Proof The cardinality of $\mathscr{B}_{n,d}$ is equal to the number of solutions $\alpha = (\alpha_\upsilon) \in \mathbb{N}^{(\mathfrak{M}^+)}$ of the system

$$\begin{cases}\sum_{\upsilon\in\mathfrak{M}^+}\alpha_\upsilon \mathrm{d}(\upsilon) = d\\ |\alpha| \leq n\end{cases} \tag{16}$$

Let α be a solution, then $|\alpha| \leq d$. Thus the number of solutions of (16) is constant for $n \geq d$. □

Let $id : F \to F$ be the identity map and define $\zeta : F \to \mathbb{K}$ by mapping x_j to 0 for $j = 1, \ldots, m$.

One can easily check that

$$id^{\otimes n-1} \otimes \zeta : F^{\otimes n} \to F^{\otimes n-1} \otimes \mathbb{K} \cong F^{\otimes n-1}$$

restricts to a surjective homomorphism of graded algebras

$$\tau_n : \mathrm{TS}^n(F) \to \mathrm{TS}^{n-1}(F) \tag{17}$$

such that

$$\begin{cases} \tau_n(e^n_\alpha) = e^{n-1}_\alpha & \text{if } |\alpha| \leq n \\ \tau_n(e^n_\alpha) = 0 & \text{if } |\alpha| = n \end{cases} \tag{18}$$

Proposition 5 *Let $d \in \mathbb{N}$ and let $\tau_{n,d} : \mathrm{TS}^n(F)_d \to \mathrm{TS}^{n-1}(F)_d$ be the restriction of τ_n to the submodule of homogeneous elements of degree d. The inverse system $(\mathrm{TS}^n(F)_d, \tau_{n,d})$ has limit $\mathscr{F}_d$ such that*

$$\mathscr{F}_d = \varprojlim_n \mathrm{TS}^n(F)_d \cong \mathrm{TS}^k(F)_d$$

for all $k \geq \sum_i d_i$.

Proof The restriction $\tau_{n,d}$ is onto for all n by (18) and Proposition 4. The result then follows by Corollary 3. □

Definition 9 Let F be $\mathbb{K}\{x_1, \ldots, x_m\}$ as usual.

1. We write
$$\mathscr{F} = \bigoplus_{d \in \mathbb{N}} \mathscr{F}_d$$
for the free $\mathbb{K}-$module direct sum of the $\mathscr{F}_d$.
2. Let $\alpha \in \mathbb{N}^{(\mathfrak{M}^+)}$, we denote by e_α the unique element of $\mathscr{F}$ corresponding to e^n_α via Proposition 5 for all $n \geq |\alpha|$.
3. For $i \in \mathbb{N} - \{0\}$ and $\upsilon \in \mathfrak{M}^+$ we denote by $e_i(\upsilon)$ the e_α having $\alpha : \mathfrak{M}^+ \to \mathbb{N}$ such that $\alpha(\upsilon) = i$ and $\alpha(\mu) = 0$ if $\mu \neq \upsilon$.
4. We denote by $\mathscr{F}^n$ for the $\mathbb{K}-$submodule generated by those e_α with $|\alpha| > n$.
5. We write $\mathscr{B} = \{e_\alpha : \alpha \in \mathbb{N}^{(\mathfrak{M}^+)}\}$ and $\mathscr{B}_d = \{e_\alpha : \partial(e_\alpha) = d\}$ for $d \in \mathbb{N}$.

It is clear the parallelism between symmetric functions and symmetric tensors over the free algebra, namely the e_α play the same role as the monomial symmetric functions play in the usual theory of symmetric functions.

Remark 12 For $d \in \mathbb{N}$ the sets $\mathscr{B}$ and $\mathscr{B}_d$ are linear bases of $\mathscr{F}$ and $\mathscr{F}_d$ respectively as it follows from Proposition 1 and Proposition 4 respectively.

For all $n \geq 1$ there are split exact sequences of $\mathbb{K}-$modules

$$0 \longrightarrow \mathscr{F}^n_d \longrightarrow \mathscr{F}_d \xrightarrow{\sigma_{n,d}} \mathrm{TS}^n(F)_d \longrightarrow 0 \tag{19}$$

and

$$0 \longrightarrow \mathscr{F}^n \longrightarrow \mathscr{F} \xrightarrow{\sigma_n} \mathrm{TS}^n(F) \longrightarrow 0 \tag{20}$$

where

$$\sigma_n = \bigoplus_{d\in\mathbb{N}} \sigma_n^d : \mathscr{F} \to \mathrm{TS}^n(F)$$

is given by

$$\sigma_n : \begin{cases} e_\alpha \mapsto e_\alpha^n, & \text{if } |\alpha| \le n \\ e_\alpha \mapsto 0, & \text{otherwise} \end{cases} \tag{21}$$

Using this splitting one can lift the product of $\mathrm{TS}^n(F)$ to $\mathscr{F}$ making it into an associative graded $\mathbb{K}-$algebra. Observe indeed that the Product Formula stabilizes for n big enough because the number of solutions of (12) is finite also if one drops out the constraint $\sum \gamma_{ij} \le n$. Thus one can express $e_\alpha^n e_\beta^n = \sum_\gamma e_\gamma^n$ with respect to $\mathscr{B}_n$ using the Product Formula with $n \gg \max(\sum_i \alpha_i, \sum \beta_j)$ and then define $e_\alpha e_\beta = \sigma_n^{-1}(\sum_\gamma e_\gamma^n) = \sum_\gamma e_\gamma$.

Proposition 6 *$\mathscr{F}$ is the inverse limit of $(TS^n(F), \tau_n)$ in the category of graded $\mathbb{K}-$algebras.*

Proof It is clear that $\mathscr{F}$ is the inverse limit of the projective system $(TS^n(F), \tau_n)$ in the category of graded $\mathbb{K}-$modules. By Proposition 5 we have $\bigcap_n \ker \sigma_n = \bigcap_n \mathscr{F}^n = \{0\}$ and the proposition is proved. □

Proposition 7 *Let $e_i(\upsilon)$ be as in Definition 9-3. The $\mathbb{K}-$algebra $\mathscr{F}$ is generated by $e_i(\upsilon)$ where $i \ge 1$ and $\upsilon \in \mathfrak{M}^+$ is primitive.*

Proof Let $n \gg |\alpha|$. By Theorem 2 e_α^n can be expressed in terms of $e_i^n(\upsilon)$ with $1 \le i \le n$ and $\upsilon \in \mathfrak{M}^+$ primitive. Using the splitting σ_n it is then possible to express any e_α as an element of the subalgebra generated by the $e_i(\upsilon)$ with $1 \le i$ and $\upsilon \in \mathfrak{M}^+$ primitive. □

Definition 10 We write $\mathfrak{e}_\alpha$ or $\mathfrak{e}_i(\upsilon)$ for $\mathfrak{ab}(e_\alpha^n)$ or $\mathfrak{ab}(e_i^n(\upsilon))$ respectively.

Corollary 4 *The $\mathbb{K}-$algebra $\mathscr{F}^{ab}$ is generated by $\mathfrak{e}_i(\upsilon)$ with $i \ge 1$ and υ varying in a complete set of representatives of Ψ.*

Proof By Proposition 7 and using the same argument in the proof of Theorem 3. □

Proposition 8 *Let $\mathscr{F}$ be endowed with the above defined product. Then the sequence (20) gives an isomorphism of graded $\mathbb{K}-$algebras*

$$\mathrm{TS}^n(F) \cong \mathscr{F}/\mathscr{F}^n.$$

Proof By construction σ_n is a surjective graded $\mathbb{K}-$algebras homomorphism whose kernel is $\mathscr{F}^n$. □

We need a Lemma.

Lemma 3 *Let A, B be two noncommutative $\mathbb{K}-$algebras and let $f : A \to B$ be a surjective homomorphism. Then*

1. *the induced homomorphism $f^{ab} : A^{ab} \to B^{ab}$ is surjective*
2. $\ker f^{ab} = \mathfrak{ab}(\ker f)$ *where $\mathfrak{ab} : A \to A^{ab}$ is the canonical homomorphism.*

Proof Since f is surjective we have that $[B,B] = f([A,A])$. The Lemma follows by the Snake Lemma and diagram chasing on the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & [A,A] & \xrightarrow{f} & [B,B] & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker f & \longrightarrow & A & \xrightarrow{f} & B & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathfrak{ab}} & & \downarrow{\scriptstyle \mathfrak{ab}} & & \downarrow{\scriptstyle \mathfrak{ab}} & & \\
0 & \longrightarrow & \ker f^{ab} & \longrightarrow & A^{ab} & \xrightarrow{f^{ab}} & B^{ab} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

□

Corollary 5 *The sequence (20) induces an isomorphism of graded $\mathbb{K}-$algebras*

$$\mathrm{TS}^n(F)^{ab} \cong \mathscr{F}^{ab}/\mathfrak{ab}(\mathscr{F}^n).$$

Proof It is enough to apply the above Lemma to $\sigma_n : \mathscr{F} \to TS^n(F)$ and then to use Proposition 8. □

5.0.1 Infinite fields

Let F^+ be the ideal of F linearly generated by the elements of $\mathfrak{M}^+$. Let $f \in \mathfrak{M}^+$ be such that $f = \sum_{\mu \in \mathfrak{M}^+} \lambda_\mu \mu$. We can express $e_k^n(f)$ in a unique way as a linear combination of e_α^n with $\alpha \in \mathbb{N}^{(\mathfrak{M}^+)}$, namely

$$e_k^n(f) = \sum_{|\alpha|=k} \lambda^\alpha e_\alpha^n \tag{22}$$

where $\lambda^\alpha = \prod_\mu \lambda_\mu^{\alpha(\mu)}$ and $n \geq k$.
We define

$$e_k(f) = \sum_{|\alpha|=k} \lambda^\alpha e_\alpha \tag{23}$$

where the right hand side is $\sigma_n^{-1}(\sum_{|\alpha|=k} \lambda^\alpha e_\alpha^n)$ for n big enough.

Proposition 9 *If $\mathbb{K}$ is an infinite field then $\mathscr{F}^n$ and $\mathfrak{ab}(\mathscr{F}^n)$ are generated as ideals by $\{e_{n+k}(f) : k \geq 1, f \in F^+\}$ and by $\{\mathfrak{e}_{n+k}(f) : k \geq 1, f \in F^+\}$ respectively.*

Proof Let $\mathscr{F}(k)$ be the subspace of $\mathscr{F}$ generated by those e_α having $|\alpha| = k$ for k a positive integer. Let $\mathscr{V}(k)$ be the subspace of $\mathscr{F}(k)$ linearly generated by $e_k(f)$ with $f \in F^+$. Suppose $\beta : \mathscr{F}(k) \to \mathbb{K}$ is a linear form that is zero on $\mathscr{V}(k)$. Then

$$\beta(e_k(f)) = \beta(\sum_{|\alpha|=k} \lambda^\alpha e_\alpha) = \sum_{|\alpha|=k} \lambda^\alpha \beta(e_\alpha) = 0$$

for all $f \in F^+$ and $k \geq 1$. Since $\mathbb{K}$ is infinite and e_α form a basis we have that β is zero on $\mathscr{F}(k)$. This means that $\mathscr{F}(k) = \mathscr{V}(k)$ and the first part of this Proposition is proved. □

5.1 Freeness

For $\mathbb{K}$ a commutative ring we shall show that $\mathscr{F}^{ab}$ is freely generated by the $\mathfrak{e}_i(\upsilon)$ where $i \geq 1$ and υ that varies in a complete set of representatives of Ψ. In order to prove this result we need some instrument coming from representations theory.

5.1.1 Generic matrices

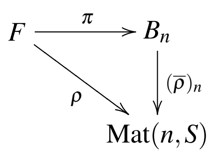
This paragraph is borrowed from C.Procesi, see [3] for a recent paper and [8] for the original source.

Let $A_n = \mathbb{K}[\xi_{hij}]$ be a polynomial ring where $i, j = 1, \ldots, n$ and $h = 1, \ldots, m$. Note that A_n is isomorphic to the symmetric $\mathbb{K}$-algebra of the dual of $\mathrm{Mat}(n, \mathbb{K})^m$.

Let F be again the free associative $\mathbb{K}$-algebra on m generators then

$$\hom_{\mathbb{K}-alg}(F, \mathrm{Mat}(n, S)) \cong \mathrm{Mat}(n, S)^m \cong \hom_{\mathbb{K}-alg}(A_n, S)$$

for any commutative $\mathbb{K}$-algebra S. More precisely set $B_n = \mathrm{Mat}(n, A_n)$ and let $\xi_h \in B_n$ be given by $(\xi_h)_{ij} = \xi_{hij}$, for all i, j, h. These are called the $n \times n$ *generic matrices* (over $\mathbb{K}$) and were introduced in the context of representation theory and rings with polynomial identities by C.Procesi (see [8]). Let $\pi_n : F \to B_n$ be the $\mathbb{K}-$algebra homomorphism given by $x_h \mapsto \xi_h$. For any $\rho \in \hom_{\mathbb{K}-alg}(F, \mathrm{Mat}(n, S))$ with S a commutative $\mathbb{K}$-algebra, there is then a unique $\overline{\rho} \in \hom_{\mathbb{K}-alg}(A_n, S)$ given by $\xi_{hij} \mapsto (\rho(\xi_h))_{ij}$ and such that the following diagram commutes



(24)

where $(\ )_n$ denotes the induced map on $n \times n$ matrices. The homomorphism π_n is called the universal $n-$dimensional representation (for the free algebra). We denote by $\mathscr{G}_n$ the subring of B_n generated by the generic matrices i.e. the image of π_n.

Definition 11 Let $C_n \subset A_n$ be the subalgebra generated by the coefficients of the characteristic polynomial of elements of $\mathscr{G}_n$. We write

$$\det(t-f) = t^n + \sum_{i=1}^{n} (-1)^i \psi_i^n(f) t^{n-i} \tag{25}$$

where $f \in \mathscr{G}_n$. Hence C_n is generated by $\psi_i^n(f)$, with $f \in F$ and $i = 1, \dots, n$.

Remark 13 The $i-$th coefficient $\psi_i^n(f)$ is the trace of $\bigwedge^i(f)$.

5.1.2 Determinant

The composition $\det \cdot \pi_n$ gives a multiplicative polynomial mapping $F \to A_n$ homogeneous of degree n hence a unique homomorphism

$$\delta_n : \mathrm{TS}^n(F) \to A_n \tag{26}$$

such that

$$\delta_n(f^{\otimes n}) = \det(\pi_n(f)) = \det(f(\xi_1, \dots, \xi_m))$$

(see [2] Prop.13 A.IV.54 and [11]).

Proposition 10 *The homomorphism of algebras (26) is a surjection onto C_n. In particular C_n is generated by $\{\psi_i^n(\pi_n(\upsilon)) : \upsilon \in \Psi\}$.*

Proof Note that $\delta_n(e_i^n(f)) = \psi_i^n(\pi_n(f))$ for all $f \in F$. The homomorphism δ_n factors through $\delta_n^{ab} : \mathrm{TS}^n(F)^{ab} \to A_n$ thus the statement follows from Theorem 3. □

We give ξ_{hij} degree $\mathrm{d}(\xi_{hij}) = 1$. Then

$$A_n = \bigoplus_{d \in \mathbb{N}} A_{n,d}$$

is a graded ring with homogeneous components $A_{n,d}$.

The homomorphism δ_n is clearly an homomorphism of graded algebras and we write

$$\delta_{n,k} : \mathrm{TS}^n(F)_k \to A_{n,k} \tag{27}$$

From Proposition 10 it follows that

$$C_n = \bigoplus_{k \in \mathbb{N}} C_{n,k}$$

where $C_{n,k} = A_{n,k} \cap C_n$ are the homogeneous component. Furthermore

$$\delta_{n,k}(\mathrm{TS}^n(F)_k) = C_{n,k} \tag{28}$$

so that δ_n is a surjective homomorphism of graded algebras.

5.1.3 Limits

For all n there is a surjective homomorphism of graded $\mathbb{K}-$algebras

$$\omega_n : A_n \to A_{n-1}$$

given by mapping ξ_{hnj} and ξ_{inh} to 0 and ξ_{hij} to ξ_{hij} for $i, j < n$.
Note that $\omega_n(\psi_i^n(f)) = \psi_i^{n-1}(f)$ if $i < n$ and $\omega_n(\psi_n^n(f)) = 0$ for any $f \in F$. Indeed the induced homomorphism $(\omega_n)_n : \mathrm{Mat}(n, A_n) \to \mathrm{Mat}(n, A_{n-1})$ is such that

$$(\omega_n)_n(\xi_h) = (\omega_n)_n(\pi_n(x_h)) = \begin{pmatrix} \pi_{n-1}(x_h) & {}^t\mathbf{0}_{n-1} \\ \mathbf{0}_{n-1} & 0 \end{pmatrix}$$

where $\mathbf{0}_{n-1}$ is the all zero row vector of $\mathbb{K}^n$ and ${}^t\mathbf{0}_{n-1}$ is the corresponding column.

Therefore we have that the restriction of ω_n to the homogeneous component $C_{n,d}$ gives a surjective $\mathbb{K}-$module homomorphism

$$\omega_{n,d} : C_{n,d} \to C_{n-1,d}$$

and the following definition makes then sense.

Definition 12 Let $d \in \mathbb{N}$, we write

$$\mathscr{C}_d = \varprojlim_n (C_{n,d}, \omega_{n,d})$$

$$\mathscr{C} = \bigoplus_{d \in \mathbb{N}} \mathscr{C}_d$$

$$\varepsilon_n = \oplus_d \varepsilon_{n,d} : \mathscr{C} \to C_n$$

where $\varepsilon_{n,d}$ is the canonical surjection $\varepsilon_{n,d} : \mathscr{C}_d \to C_{n,d}$.

For $\upsilon \in \mathfrak{M}^+$

$$\psi_i(\upsilon) = \varprojlim_n \psi_i^n(\upsilon)$$

Proposition 11 *The ring $\mathscr{C}$ is a polynomial ring freely generated by the $\psi_i(\upsilon)$ with $i \geq 1$ and υ that varies in a complete set of representatives of Ψ. The ε_n are homomorphism of graded algebras and $\mathscr{C}$ is the inverse limit of the projective system (C_n, ω_n) in the category of graded $\mathbb{K}-$algebras.*

Proof By §3, (10) in [5] , the ending Remark in [5] and Complements in [4]. □

Lemma 4 *There is a unique surjective homomorphism of graded $\mathbb{K}-$algebras $\delta : \mathscr{F} \to \mathscr{C}$ such that $\varepsilon_n \delta = \delta_n \sigma_n$ for all n.*

Proof The following diagram commutes in the category of graded ring and all its arrows are surjections

$$\begin{array}{ccc} & \mathscr{F} = \bigoplus_{d \in \mathbb{N}} \mathscr{F}_d & \\ \swarrow^{\delta_n \sigma_n} & & \searrow^{\delta_{n-1} \sigma_{n-1}} \\ C_n = \bigoplus_d C_{n,d} & \xrightarrow{\omega_n = \oplus_d \omega_{n,d}} & C_{n-1} = \bigoplus_d C_{n-1,d} \end{array}$$

By Proposition 11 we have a unique homomorphism of graded $\mathbb{K}-$algebra $\delta : \mathscr{F} \to C$ such that $\delta(e_i(\upsilon)) = \psi_i(\upsilon)$ for all $i \in \mathbb{N}$ and $\upsilon \in \mathfrak{M}^+$. From Propositions 7 and 11 it follows that δ is onto. □

We restate here Theorem 1.1. in a more precise way

Theorem 4 *The algebra $\mathscr{F}^{ab}$ is a free polynomial ring freely generated by $\mathfrak{e}_i(\upsilon)$ with $i \geq 1$ and υ that varies in a complete set of representatives of Ψ. It is isomorphic to $\mathscr{C}$ through $\mathfrak{e}_i(\upsilon) \leftrightarrow \psi_i(\upsilon)$.*

Proof Since $\mathscr{C}$ is free we have the homomorphism of commutative graded algebras $\mathscr{C} \to \mathscr{F}^{ab}$ given by $\psi_i(\upsilon) \mapsto \mathfrak{e}_i(\upsilon)$. By the previous Lemma this is the inverse of the one $\mathscr{F}^{ab} \to \mathscr{C}$ induced by δ. □

Remark 14 D.Ziplies has introduced the gamma algebra $\hat{\Gamma}(F^+)$ in [16]. The above Proposition, although new, can also be proved observing that $\mathscr{F} \cong \hat{\Gamma}(F^+)$ and applying then Th.4.4 [17].

6 Invariants of several matrices

For $n \in \mathbb{N}$ we have a commutative diagram

$$\begin{array}{ccc} \mathscr{F}^{ab} & \xrightarrow{\sigma_n^{ab}} & \mathrm{TS}^n(F)^{ab} \\ \downarrow \cong & & \downarrow \delta_n^{ab} \\ \mathscr{C} & \xrightarrow[\varepsilon_n]{} & C_n \end{array}$$

we shall show in this last chapter that δ_n^{ab} is an isomorphism when $\mathbb{K}$ is an infinite field or the ring $\mathbb{Z}$ of integers.

6.1 Matrix Invariant

The general linear group $G = \mathrm{GL}(n,\mathbb{K})$ of $n \times n$ invertible matrices of $\mathrm{Mat}(n,\mathbb{K})$ acts on $\mathrm{Mat}(n,\mathbb{K})^m$ by simultaneous conjugation, i.e. via basis change on $\mathbb{K}^n$. The $\mathbb{K}-$algebra $A_n = \mathbb{K}[\xi_{hij}]$ is isomorphic to the symmetric algebra on the dual of $\mathrm{Mat}(n,\mathbb{K})^m$ so that the above action induces another on A_n and we denote by A_n^G the subalgebra of the invariants for this action. Let $\mathscr{M}_n^m$ denote $\mathrm{Spec} A_n$. The categorical quotient $\mathscr{M}_n^m//G$ is defined as

$$\mathscr{M}_n^m//G = \mathrm{Spec} A_n^G \tag{29}$$

The affine scheme $\mathscr{M}_n^m//G$ is the coarse moduli space parameterizing the $n-$dimensional linear representations of F up to base change and its geometric points correspond to the orbits of the semi-simple representations of F. We refer the reader to [1,4,8] for masterpieces on this subject. Recall that the ring C_n is generated by the coefficients $\psi_k(f)$ of characteristic polynomial

$$\det(t - \pi_n(f)) = t^n + \sum_{i=1}^{n} (-1)^i \psi_i(f) t^{n-i}$$

where $f \in F$.

Being the determinant invariant under base change we have that the ring C_n is made of invariants i.e $C_n \subset A_n^G$. When $\mathbb{K}$ is a characteristic zero field it was showed by C.Procesi [8] and separately K.S.Sibirskiĭ [13] that $C_n = A_n^G$. This has been generalized to the positive characteristic case by S. Donkin and then by A. Zubkov by proving a conjecture of C.Procesi, see [4,8,18]. The celebrated Procesi-Razmyslov theorem [9,10] can be reformulated by saying that the kernel of the surjection

$$\varepsilon_n : \mathscr{C} \to C_n = A_n^G \tag{30}$$

is generated by $\psi_{n+1}(f)$ and $f \in F^+$. When $\mathbb{K}$ is an infinite field Zubkov gave in [19] a generalization of Procesi-Razmyslov theorem by proving that the kernel of the surjection (30) is the ideal

$$< \{\psi_i(f) : i > n, f \in F^+\} > \tag{31}$$

Theorem 5 *Let $\mathbb{K}$ be an infinite field or the ring $\mathbb{Z}$ of integers. The homomorphism*

$$\delta_n^{ab} : \mathrm{TS}^n(F)^{ab} \to C_n$$

induced by the composition $\det \cdot \pi_n$ *of the determinant with the universal representation is an isomorphism.*

Proof After Theorem 4 we have that Corollary 5 gives a presentation of $\mathrm{TS}^n(F)^{ab}$ as a quotient of a polynomial ring, namely

$$\mathfrak{ab}(\mathscr{F}^n) \longrightarrow \mathbb{K}[\{\mathfrak{e}_i(\upsilon) : i \geq 1, \upsilon \in \Psi\}] \xrightarrow{\sigma_n^{ab}} \mathrm{TS}^n(F)^{ab}$$

When $\mathbb{K}$ is an infinite field we have that

$$\begin{aligned} \mathrm{TS}^n(F)^{ab} &\cong \mathbb{K}[\{\mathfrak{e}_i(\upsilon) : i \geq 1, \upsilon \in \Psi\}]/ < \{\mathfrak{e}_i(f) : i > n, f \in F^+\} > \\ &\cong \mathscr{C}/ < \{\psi_i(f) : i > n, f \in F^+\} > \end{aligned}$$

thus the result follows from (30) and (31).

Let now $\mathbb{K}$ be an arbitrary commutative ring. We denote by F the free algebra with coefficients in $\mathbb{Z}$ and by $F_{\mathbb{K}} \cong \mathbb{K} \otimes_{\mathbb{Z}} F$ the one with coefficients in $\mathbb{K}$.

For a commutative ring $\mathbb{K}$ one has $\mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}}) \cong \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{TS}^n_{\mathbb{Z}}(F)$ thus the homomorphism

$$id_{\mathbb{K}} \otimes \mathfrak{ab}_{\mathbb{Z}} : \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{TS}^n_{\mathbb{Z}}(F) \to \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{TS}^n_{\mathbb{Z}}(F)^{ab}$$

factors through $\mathfrak{ab}_{\mathbb{K}} : \mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}}) \to \mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}})^{ab}$ and the following diagram commutes

$$\begin{array}{ccc} \mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}}) & \xrightarrow{\cong} & \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{TS}^n_{\mathbb{Z}}(F) \\ \downarrow{\scriptstyle \mathfrak{ab}_{\mathbb{K}}} & & \downarrow{\scriptstyle id_{\mathbb{K}} \otimes \mathfrak{ab}_{\mathbb{Z}}} \\ \mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}})^{ab} & \longrightarrow & \mathbb{K} \otimes_{\mathbb{Z}} \mathrm{TS}^n_{\mathbb{Z}}(F)^{ab} \end{array} \tag{32}$$

In [4] it is also proved that $C_n \subset A_n = \mathbb{Z}[\xi_{hij}]$ is a $\mathbb{Z}-$form of the ring of invariants, i.e. $\mathbb{F} \otimes_{\mathbb{Z}} C_n \cong \mathbb{F}[\mathrm{Mat}(n,\mathbb{F})^m]^{\mathrm{GL}(n,\mathbb{F})}$ for all algebraically closed field $\mathbb{F}$. Let $\mathbb{K}$ be an algebraically closed field. We have the surjection $\delta_n^{ab} : \mathrm{TS}^n_{\mathbb{Z}}(F)^{ab} \to C_n$

induced by the composition of the universal $n-$dimensional representation with the determinant.

$$\begin{array}{ccc}
\mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}}) & \xrightarrow{\cong} & \mathbb{K}\otimes_{\mathbb{Z}}\mathrm{TS}^n_{\mathbb{Z}}(F) \\
\downarrow{\scriptstyle \mathfrak{ab}_{\mathbb{K}}} & & \downarrow{\scriptstyle id_{\mathbb{K}}\otimes\mathfrak{ab}_{\mathbb{Z}}} \\
\mathrm{TS}^n_{\mathbb{K}}(F_{\mathbb{K}})^{ab} & \longrightarrow & \mathbb{K}\otimes_{\mathbb{Z}}\mathrm{TS}^n_{\mathbb{Z}}(F)^{ab} \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle id_{\mathbb{K}}\otimes\delta_n^{ab}} \\
\mathbb{K}[\mathrm{Mat}(n,\mathbb{K})^m]^{\mathrm{GL}(n,\mathbb{K})} & \xrightarrow{\cong} & \mathbb{K}\otimes_{\mathbb{Z}} C_n
\end{array} \tag{33}$$

For algebraically closed fields we then have

$$\mathbb{K}\otimes_{\mathbb{Z}}\mathrm{TS}^n_{\mathbb{Z}}(F)^{ab}\cong\mathbb{K}\otimes_{\mathbb{Z}} C_n$$

as graded rings with finitely generated homogenous summands and the result follows. □

We are now able to proof Theorem 1.

Proof (Theorem 1) It follows from Theorem 5 and (30), (31).

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