Algorithms for computing spectral decomposition

Niigata University
Yukie Iizuka

plan

- background
- residue of resolvent
- algorithms
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    - irreducible
    - factorization
  - minimal polynomial : not square free
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    - factorization
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background

- There are no algorithm for exactly computing spectral decomposition (as far as we know).
- You use resolvent $\left(= (\lambda E - A)^{-1}\right)$ directly.

$\rightarrow$ The cost of computing inverse matrix

is high.

We propose another method which is

free from computing inverse matrix.

spectral decomposition of matrix

A : $n \times n$ square matrix , E : identity matrix

$\pi(\lambda)$: minimal polynomial

α_j $(j = 1, 2, \ldots, t \mid t \leq n)$: eigenvalues

$$\begin{cases} \sum_{j=1}^{t} P(\alpha_j) = E \\ \sum_{j=1}^{t} \left(\alpha_j P(\alpha_j) + D(\alpha_j)\right) = A \end{cases}$$

spectral decomposition of matrix

spectral decomposition of matrix

$P(\alpha_j)$: coefficient matrix of

the pole of order 1 at α_j

$D(\alpha_j)$: coefficient matrix of

the pole of order 2 at α_j

residue of resolvent

define polynomial $q(x, y)$ by

$$\pi(x) - \pi(y) = q(x, y)(x - y)$$

substitute A for x and λE for y

$$\underbrace{\pi(A)}_{0} - \underbrace{\pi(\lambda E)}_{\pi(\lambda)E} = q(A, \lambda E)(A - \lambda E)$$

$$\pi(\lambda)E = q(A, \lambda E)(\lambda E - A)$$

$$(\lambda E - A)^{-1} = \frac{1}{\pi(\lambda)} q(A, \lambda E)$$

residue of resolvent

Moreover, we set $\psi(A, \lambda) = q(A, \lambda E)$.

$$(\lambda E - A)^{-1} = \frac{\psi(A, \lambda)}{\pi(\lambda)}$$

This is free from inverse matrix.

minimal polynomial : square free

We consider only the pole of order 1. $\longrightarrow P(\alpha_j)$

$$P(\alpha_j) = \frac{1}{2\pi i}\oint_{\alpha_j} (\lambda E - A)^{-1} d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{1}{\pi(\lambda)}\psi(A,\lambda)d\lambda$$

$\pi(\lambda)$ *and* $\pi'(\lambda)$ *: prime each other*

$$a(\lambda)\pi(\lambda) + b(\lambda)\pi'(\lambda) = 1$$

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{a(\lambda)\pi(\lambda) + b(\lambda)\pi'(\lambda)}{\pi(\lambda)}\psi(A,\lambda)d\lambda$$

singularity

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{\pi'(\lambda)}{\pi(\lambda)} b(\lambda)\psi(A,\lambda)d\lambda$$

$$\frac{1}{2\pi i}\oint_{\alpha} \frac{\pi'(x)}{\pi(x)} h(x)dx = h(\alpha)$$

$$= b(\alpha_j)\,\psi(A,\alpha_j)$$

minimal polynomial : square free

To compute fast …

Theorem.

If a matrix A is diagonalizable then the projection matrix $P(\lambda)$ has the following representation :

$$P(\lambda) = b(\lambda)\ \psi(A, \lambda) = b(A)\ \psi(A, \lambda)\ .$$

We compute by using this theorem.

minimal polynomial : square free

$$\pi(\lambda) = p_1(\lambda) p_2(\lambda) \cdots p_s(\lambda)$$

$\longrightarrow$ We can compute for each factor.

example : focus on $p_1(\lambda)$.

$$\psi_1(x, y) = \frac{p_1(x) - p_1(y)}{x - y}$$

$$h_1(\lambda) = p_2(\lambda) \cdots p_s(\lambda)$$

minimal polynomial : square free

$\pi(x)$ $\pi(\lambda)$

$$\psi(x,\lambda) = \frac{p_1(x)h_1(x) - p_1(\lambda)h_1(\lambda)}{x-\lambda}$$

$$= \frac{p_1(x)h_1(x) - p_1(\lambda)h_1(x) + p_1(\lambda)h_1(x) - p_1(\lambda)h_1(\lambda)}{x-\lambda}$$

$$= \frac{p_1(x) - p_1(\lambda)}{x-\lambda} h_1(x) + \frac{h_1(x) - h_1(\lambda)}{x-\lambda} p_1(\lambda)$$

$$= \psi_1(x,\lambda)h_1(x) + \frac{h_1(x) - h_1(\lambda)}{x-\lambda} p_1(\lambda)$$

minimal polynomial : square free

$$P(\alpha_j) = \frac{1}{2\pi i}\oint_{\alpha_j} (\lambda E - A)^{-1} d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{1}{\pi(\lambda)}\psi(A,\lambda)d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{a_1(\lambda)p_1(\lambda) + b_1(\lambda)p_1'(\lambda)h_1(\lambda)}{p_1(\lambda)h_1(\lambda)}\left\{\psi_1(A,\lambda)h_1(A) + \frac{h_1(A) - h_1(\lambda E)}{A - \lambda E}p_1(\lambda)\right\}d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\alpha_j} \frac{p_1'(\lambda)}{p_1(\lambda)} b_1(\lambda)\psi_1(A,\lambda)h_1(A)d\lambda$$

$$= b_1(\alpha_j)\,\psi_1(A,\alpha_j)h_1(A)$$

$$\boxed{P(\lambda) = b_1(A)\,\psi_1(A,\lambda)h_1(A)}$$

minimal polynomial : not square free

compute Laurent expansion of resolvent

at $\lambda = \alpha_j$ (l_j : multiplicity)

$$\left(\lambda E - A\right)^{-1} = \frac{P(\alpha_j)}{\lambda - \alpha_j} + \sum_{k=1}^{l_j - 1} \frac{D(\alpha_j)^k}{\left(\lambda - \alpha_j\right)^{k+1}} + \left(regular\ terms\ \ for\ \lambda\right)$$

We consider the poles of order 1 and 2 .

$\longrightarrow$ $P(\alpha_j), D(\alpha_j)$

minimal polynomial : not square free

- preparation

We set $p(\alpha) = 0$, $p(\lambda) = (\lambda - \alpha)\, g(\lambda)$.
differentiate both terms at λ

$$p'(\lambda) = g(\lambda) + (\lambda - \alpha)g'(\lambda)$$
$$p'(\alpha) = g(\alpha)$$

repeat differentiation

$$p^{(m)}(\alpha) = mg^{(m-1)}(\alpha)$$

minimal polynomial : not square free

$$\pi(\lambda) = p(\lambda)^l :$$

$$\frac{1}{2\pi i}\oint_{\lambda=\alpha} (\lambda E - A)^{-1} d\lambda = \frac{1}{2\pi i}\oint_{\lambda=\alpha} \frac{\psi(A,\lambda)}{\pi(\lambda)} d\lambda = \frac{1}{2\pi i}\oint_{\lambda=\alpha} \frac{\psi(A,\lambda)}{p(\lambda)^l} d\lambda$$

$$= \frac{1}{2\pi i}\oint_{\lambda=\alpha} \frac{1}{(\lambda-\alpha)^l} \left(\frac{\psi(A,\lambda)}{g(\lambda)^l} \right) d\lambda$$

Taylor expansion

$$\frac{\psi(x,\lambda)}{g(\lambda)^l} = c_0 + c_1(\lambda-\alpha) + c_2(\lambda-\alpha)^2 + \cdots$$

minimal polynomial : not square free

$$\frac{\psi(x,\lambda)}{p(\lambda)^l} = \frac{1}{(\lambda-\alpha)^l}\left(c_0 + c_1(\lambda-\alpha) + c_2(\lambda-\alpha)^2 + \cdots\right)$$

$$= \frac{c_0}{(\lambda-\alpha)^l} + \cdots + \frac{c_{l-2}}{(\lambda-\alpha)^2} + \frac{c_{l-1}}{\lambda-\alpha} + (\textit{regular terms for } \lambda)$$

the pole of order 2 the pole of order 1

calculating c_{l-2}, c_{l-1}

$\longrightarrow$ We can express

spectral decomposition.

minimal polynomial : not square free

- example : multiplicity $l = 2 \longrightarrow c_0, c_1$

preparation

- $p^{(m)}(\alpha) = mg^{(m-1)}(\alpha)$
- $\dfrac{1}{p'(\alpha)} = b(\alpha)$

$\because a(\lambda)p(\lambda) + b(\lambda)p'(\lambda) = 1 \ , \ p(\alpha) = 0$

minimal polynomial : not square free

$$\frac{\psi(x,\lambda)}{g(\lambda)^2} = c_0 + c_1(\lambda-\alpha) + c_2(\lambda-\alpha)^2 + \cdots \qquad \cdots (\#)$$

substitute α for λ in (#)

$$c_0 = \frac{\psi(x,\alpha)}{g(\alpha)^2} = \frac{\psi(x,\alpha)}{p'(\alpha)^2} = \psi(x,\alpha)b(\alpha)^2$$

differentiate both terms of (#) at λ

$$\left(\frac{\psi(x,\lambda)}{g(\lambda)^2}\right)' = \frac{\psi'(x,\lambda)g(\lambda) - 2\psi(x,\lambda)g'(\lambda)}{g(\lambda)^3}$$

$$= c_1 + 2c_2(\lambda-\alpha) + 3c_3(\lambda-\alpha)^2 + \cdots \qquad \cdots (\#\#)$$

minimal polynomial : not square free

substitute α for λ in (##)

$p''(\alpha)$

$$c_1 = \frac{\psi'(x,\alpha)g(\alpha) - \psi(x,\alpha)\cdot 2g'(\alpha)}{g(\alpha)^3} = \frac{\psi'(x,\alpha)p'(\alpha) - \psi(x,\alpha)p''(\alpha)}{p'(\alpha)^3}$$
$$= \left(\psi'(x,\alpha)p'(\alpha) - \psi(x,\alpha)p''(\alpha)\right)b(\alpha)^3$$

$$c_0 = \psi(x,\alpha)b(\alpha)^2$$
$$c_1 = \left(\psi'(x,\alpha)p'(\alpha) - \psi(x,\alpha)p''(\alpha)\right)b(\alpha)^3$$

These are common expressions for eigenvalue α.
So we replace eigenvalue α with variable λ.

minimal polynomial : not square free

Because of $p(\alpha) = 0$, we set

$$c_0(x,\lambda) = \left(\psi(x,\lambda) b(\lambda)^2 \right) \bmod p(\lambda)$$

$$c_1(x,\lambda) = \left\{ \left(\psi'(x,\lambda) p'(\lambda) - \psi(x,\lambda) p''(\lambda) \right) b(\lambda)^3 \right\} \bmod p(\lambda) .$$

$c_0(A, \lambda)$: coefficient matrix of the pole of order 2 at α_j $\longrightarrow$ $D(\lambda)$

$c_1(A, \lambda)$: coefficient matrix of the pole of order 1 at α_j $\longrightarrow$ $P(\lambda)$

minimal polynomial : not square free

$$\pi(\lambda) = p_1(\lambda)^{l_1}\, p_2(\lambda)^{l_2} \cdots p_s(\lambda)^{l_s}$$

$\longrightarrow$ We can compute for each factor, too.

example : focus on $p_1(\lambda)$.

$$\tilde{\psi}_1(x,y) = \frac{p_1(x)^{l_1} - p_1(y)^{l_1}}{x-y}$$

$$\tilde{h}_1(\lambda) = p_2(\lambda)^{l_2} \cdots p_s(\lambda)^{l_s}$$

minimal polynomial : not square free

$\pi(x)$ $\quad$ $\pi(\lambda)$

$$\psi(x,\lambda) = \frac{p_1(x)^{l_1}\tilde{h}_1(x) - p_1(\lambda)^{l_1}\tilde{h}_1(\lambda)}{x-\lambda}$$

$$= \frac{p_1(x)^{l_1}\tilde{h}_1(x) - p_1(\lambda)^{l_1}\tilde{h}_1(x) + p_1(\lambda)^{l_1}\tilde{h}_1(x) - p_1(\lambda)^{l_1}\tilde{h}_1(\lambda)}{x-\lambda}$$

$$= \frac{p_1(x)^{l_1} - p_1(\lambda)^{l_1}}{x-\lambda}\tilde{h}_1(x) + \frac{\tilde{h}_1(x) - \tilde{h}_1(\lambda)}{x-\lambda}p_1(\lambda)^{l_1}$$

$$= \tilde{\psi}_1(x,\lambda)\tilde{h}_1(x) + \frac{\tilde{h}_1(x) - \tilde{h}_1(\lambda)}{x-\lambda}p_1(\lambda)^{l_1}$$

minimal polynomial : not square free

$$\frac{\psi(x,\lambda)}{\pi(\lambda)} = \frac{1}{p_1(\lambda)^{l_1}\tilde{h}_1(\lambda)}\left\{\tilde{\psi}_1(x,\lambda)\tilde{h}_1(x) + \frac{\tilde{h}_1(x)-\tilde{h}_1(\lambda)}{x-\lambda}p_1(\lambda)^{l_1}\right\}$$

$$= \boxed{\frac{1}{p_1(\lambda)^{l_1}}\cdot\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{h}_1(\lambda)}\cdot\tilde{h}_1(x)} + \boxed{\frac{1}{\tilde{h}_1(\lambda)}\cdot\frac{\tilde{h}_1(x)-\tilde{h}_1(\lambda)}{x-\lambda}}$$

singularity at $\lambda = \alpha$

We consider Laurent expansion of

<u>the first term of right side</u> at $\lambda = \alpha$.

$$\frac{1}{p_1(\lambda)^{l_1}}\cdot\frac{\tilde{\psi}_1(x,\lambda)}{\tilde{h}_1(\lambda)}\cdot\tilde{h}_1(x)$$

matrix (substitute A for x)

multiply $h_1(A)$ after compute here

minimal polynomial : not square free

We set $p_1(\lambda) = (\lambda - \alpha)\, g_1(\lambda)$.

$$\frac{1}{p_1(\lambda)^{l_1}} \cdot \frac{\tilde{\psi}_1(x,\lambda)}{\tilde{h}_1(\lambda)} = \frac{1}{(\lambda-\alpha)^{l_1}} \cdot \underbrace{\boxed{\frac{\tilde{\psi}_1(x,\lambda)}{g_1(\lambda)^{l_1}\,\tilde{h}_1(\lambda)}}}_{r_1(\lambda)}$$

And we consider Taylor expansion of $r_1(\lambda)$.

$$r_1(\lambda) = r_1(\alpha) + r_1'(\alpha)(\lambda-\alpha) + \frac{1}{2!} r_1''(\alpha)(\lambda-\alpha)^2 + \frac{1}{3!} r_1'''(\alpha)(\lambda-\alpha)^3 + \cdots$$

$$\frac{r_1(\lambda)}{(\lambda-\alpha)^{l_1}} = \frac{r_1(\alpha)}{(\lambda-\alpha)^{l_1}} + \cdots + \frac{1}{(l_1-2)!}\frac{r_1^{(l_1-2)}(\alpha)}{(\lambda-\alpha)^2} + \frac{1}{(l_1-1)!}\frac{r_1^{(l_1-1)}(\alpha)}{\lambda-\alpha} + (\textit{regular terms for } \lambda)$$

the pole of order 2 (the term $\frac{1}{(l_1-2)!}\frac{r_1^{(l_1-2)}(\alpha)}{(\lambda-\alpha)^2}$)

the pole of order 1 (the term $\frac{1}{(l_1-1)!}\frac{r_1^{(l_1-1)}(\alpha)}{\lambda-\alpha}$)

minimal polynomial : not square free

$$r_1^{(l_1-2)}(x,\lambda) = r_1^{(l_1-2)}(\lambda) \bmod p_1(\lambda)$$

$$r_1^{(l_1-1)}(x,\lambda) = r_1^{(l_1-1)}(\lambda) \bmod p_1(\lambda)$$

Because of $p_1(\alpha) = 0$,

it consists of $r_1(\alpha) = r_1(x,\alpha)$, $r_1'(\alpha) = r_1'(x,\alpha)$.

minimal polynomial : not square free

Spectral decompositions at $\lambda = \alpha_{ki}$ are as follows.

$$P(\alpha_{ki}) = \frac{1}{(l_k - 1)!} h_k(A) r_k^{(l_k - 1)}(A, \alpha_{ki})$$

$$D(\alpha_{ki}) = \frac{1}{(l_k - 2)!} h_k(A) r_k^{(l_k - 2)}(A, \alpha_{ki})$$

$P(\alpha_{ki})$, $D(\alpha_{ki})$ satisfy as follows.

$$\sum_k \left(\sum_i P(\alpha_{ki}) \right) = E \ , \ \sum_k \left\{ \sum_i \left(\alpha_{ki} P(\alpha_{ki}) + D(\alpha_{ki}) \right) \right\} = A$$

example

For

$$A=\begin{pmatrix}-4 & 5 & -4\\ -3 & 3 & 3\\ -1 & -2 & 3\end{pmatrix},\ E=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix},\ f(\lambda)=\lambda^3-2\lambda^2+2\lambda+66\ ,$$

compute spectral decomposition.

$$b(\lambda)=\frac{1}{60134}(2\lambda^2-303\lambda+202)$$

$$\psi(A,\lambda)=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\lambda^2+\begin{pmatrix}-6 & 5 & -4\\ -3 & 1 & 3\\ -1 & -2 & 1\end{pmatrix}\lambda+\begin{pmatrix}15 & -7 & 27\\ 6 & -16 & 24\\ 9 & -13 & 3\end{pmatrix}$$

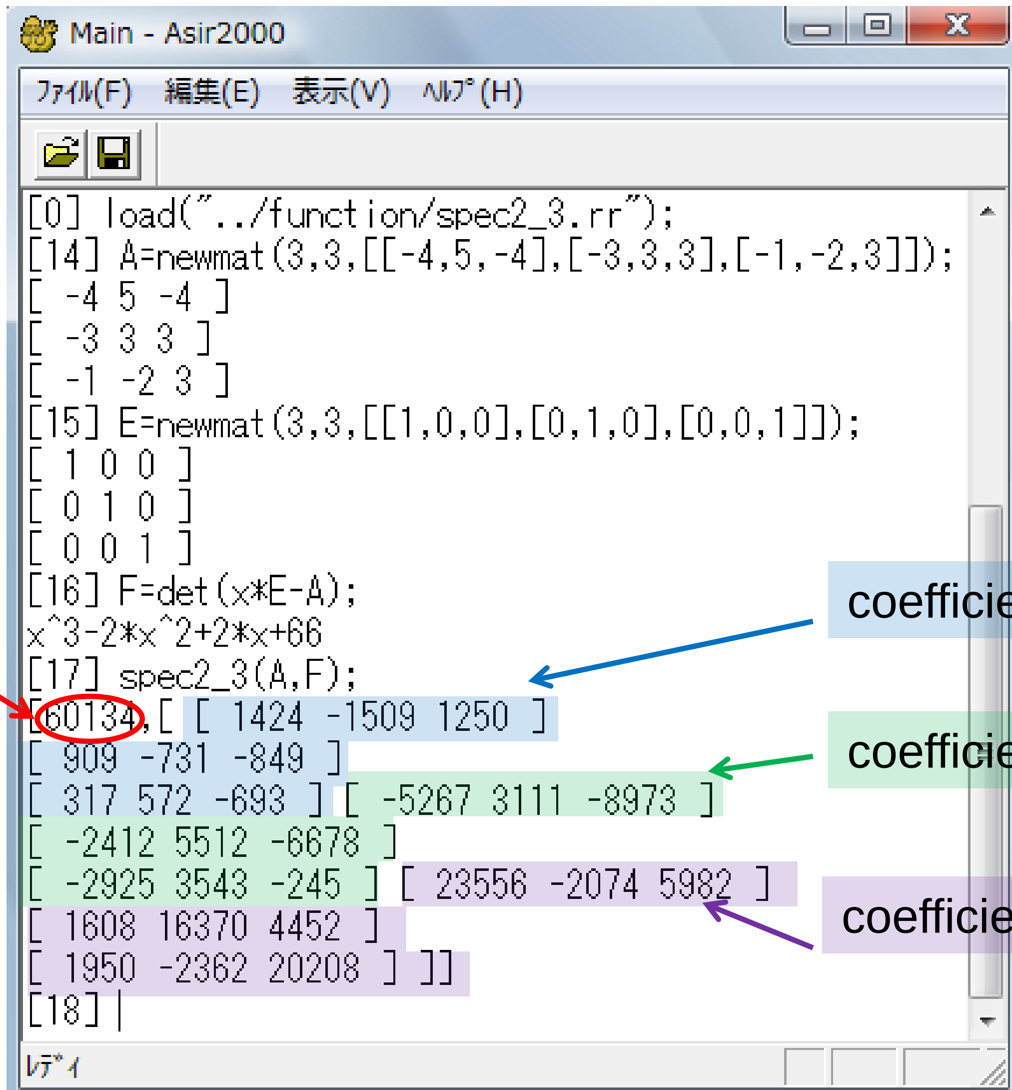
example

output :

$$P(\lambda) = \frac{1}{60134}\left(\begin{pmatrix} 1424 & -1509 & 1250 \\ 909 & -731 & -849 \\ 317 & 572 & -693 \end{pmatrix}\lambda^2 + \begin{pmatrix} -5267 & 3111 & -8973 \\ -2412 & 5512 & -6678 \\ -2925 & 3543 & -245 \end{pmatrix}\lambda + \begin{pmatrix} 23556 & -2074 & 5982 \\ 1608 & 16370 & 4452 \\ 1950 & -2362 & 20208 \end{pmatrix}\right)$$

If you substitute α, β, γ for λ ,
you get the spectral decomposition.

example


Main - Asir2000
ﾌｧｲﾙ(F) 編集(E) 表示(V) ﾍﾙﾌﾟ(H)
[0] load("../function/spec2_3.rr");
[14] A=newmat(3,3,[[-4,5,-4],[-3,3,3],[-1,-2,3]]);
[-4 5 -4]
[-3 3 3]
[-1 -2 3]
[15] E=newmat(3,3,[[1,0,0],[0,1,0],[0,0,1]]);
[1 0 0]
[0 1 0]
[0 0 1]
[16] F=det(x*E-A);
x^3-2*x^2+2*x+66
[17] spec2_3(A,F);
[60134,[[1424 -1509 1250]
[909 -731 -849]
[317 572 -693] [-5267 3111 -8973]
[-2412 5512 -6678]
[-2925 3543 -245] [23556 -2074 5982]
[1608 16370 4452]
[1950 -2362 20208]]]
[18]
ﾚﾃﾞｨ
denominator
coefficient matrix of λ²
coefficient matrix of λ
coefficient matrix of constant

assignment

- We are trying to make an algorithm to compute efficiently minimal polynomial.
  - We want to combine it with spectral decomposition algorithms.
- We compute a column of spectral decomposition and parallel computation.

Thank you for your attention.