The Riemann hypothesis and Physics

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A mathematical problem

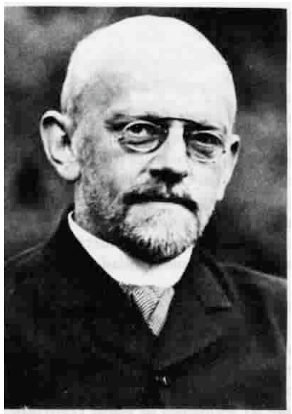
Riemann hypothesis (1859):

the complex zeros of the classical zeta function $\varsigma(s)$ all have real part equal to 1/2

$$\varsigma(s_n) = 0, s_n \in C \to s_n = \frac{1}{2} + iE_n, \quad E_n \in \Re, n \in Z$$

Long history:

- 8th Hilbert problem (1900)
- 1st Smale problem (2000)
- Millennium problem (2000)



A physical approach

Polya and Hilbert conjecture (circa 1910):

There exists a self-adjoint operator H whose discrete spectra is given by the imaginary part of the Riemann zeros,

$$H\left|\psi_n\right\rangle = E_n\left|\psi_n\right\rangle \Rightarrow E_n \in \Re \Rightarrow RH : True$$

This is known as the spectral approach to the RH

The problem is to find H: the Riemann operator

Outline

- The Riemann zeta function
- Three hints for an spectral interpretation of Riemann zeros
- The Berry-Keating/Connes model
- Landau version of the H = xp model

Based on:

"Landau levels and Riemann zeros"
G.S. and P.K.Townsend
Physical Review Letters 2008 (arXive:0805.4079)

"A quantum mechanical model of the Riemann zeros"
G.S. New Journal of Physics 2008 (arXive:0712.0705)

The Riemann zeta function

The zeta function: Rosetta Stone in Maths

Zeta(s) can be written in three different "languages"

Sum over the integers (Euler)

$$\varsigma(s) = \sum_{1}^{\infty} \frac{1}{n^s}\,, \ \mathrm{Re}\, s > 1$$

Product over primes (Euler)

$$\varsigma(s) = \prod_{p=2,3,5,\ldots} \frac{1}{1-p^{-s}}\,, \mathrm{Re}\, s > 1$$

Product over the zeros (Riemann)

$$\varsigma(s) = \frac{\pi^{s/2}}{2(s-1)\Gamma(1+s/2)} \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

Order in the prime numbers

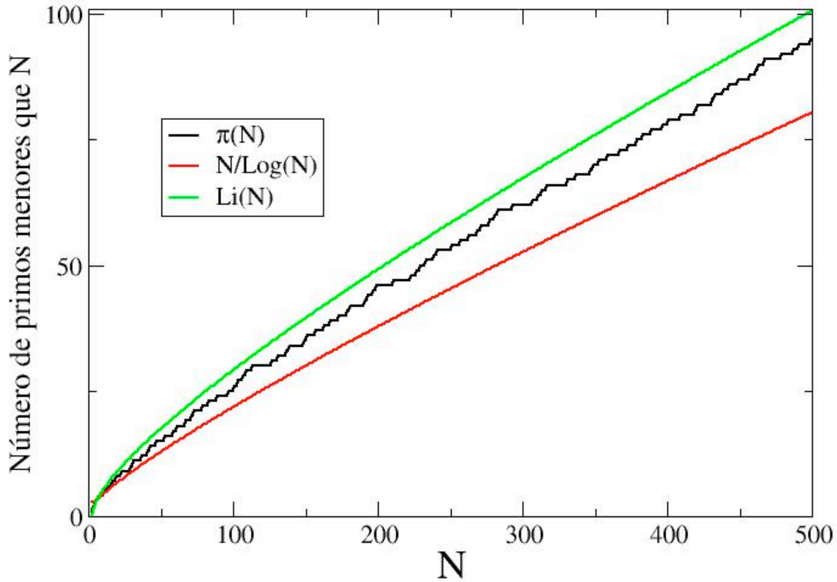
$\pi(x):$ Number of primes less than x

e.g. $\pi(100) = 25$

The asymptotic distribution of primes is given by Gauss law:

$$\pi(x) \approx \frac{x}{\log x}, \quad x \to \infty$$

Prime number theorem
Hadamard - Vallée-Poussin (1896)



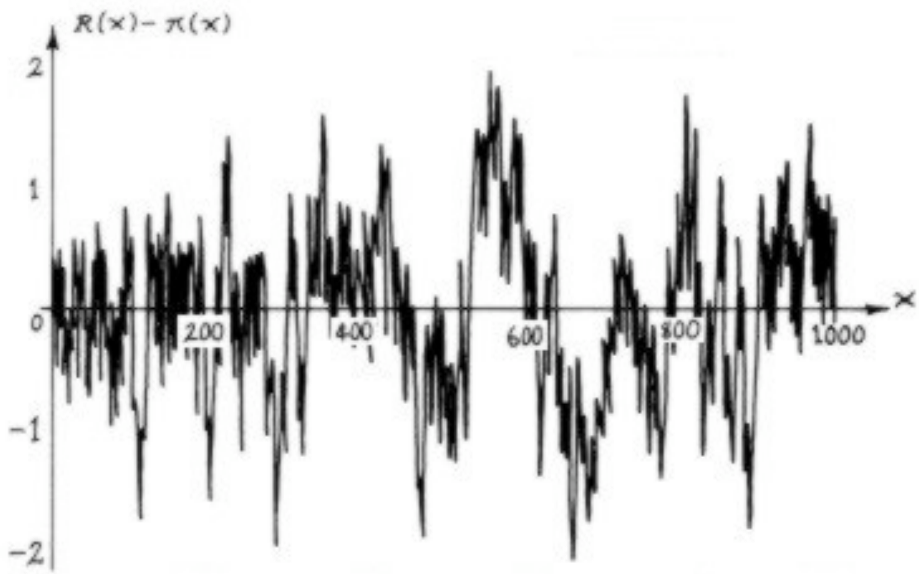
$$Li(x) = \int_2^x \frac{dt}{\log t}$$

Disorder in the prime numbers

The prime numbers appear almost at random

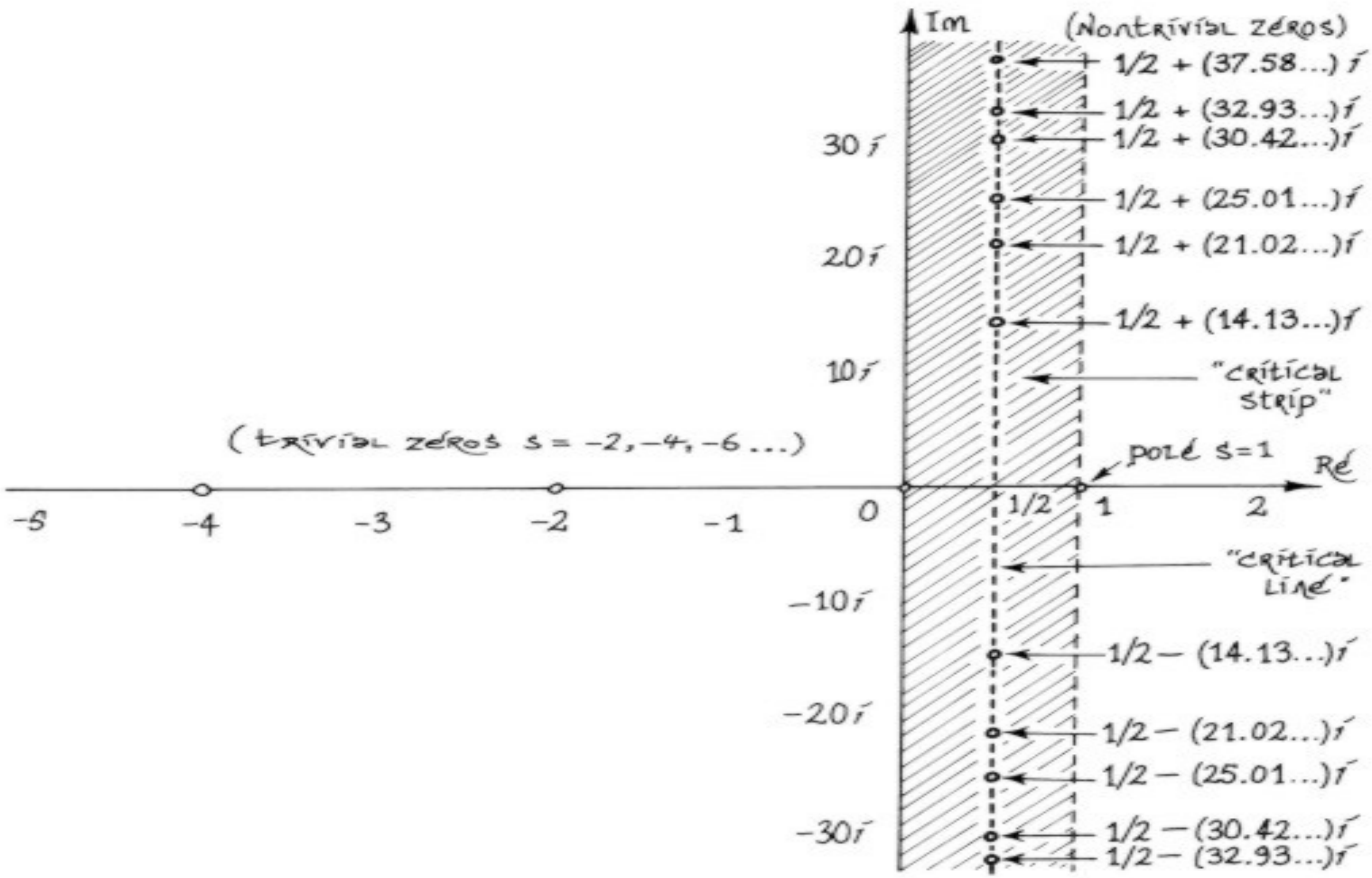
The RH puts a bound to the deviation of the Gauss law:

$$RH : true \Leftrightarrow \left|\pi(x) - Li(x)\right| \leq c\sqrt{x}\log x, \quad x \to \infty$$



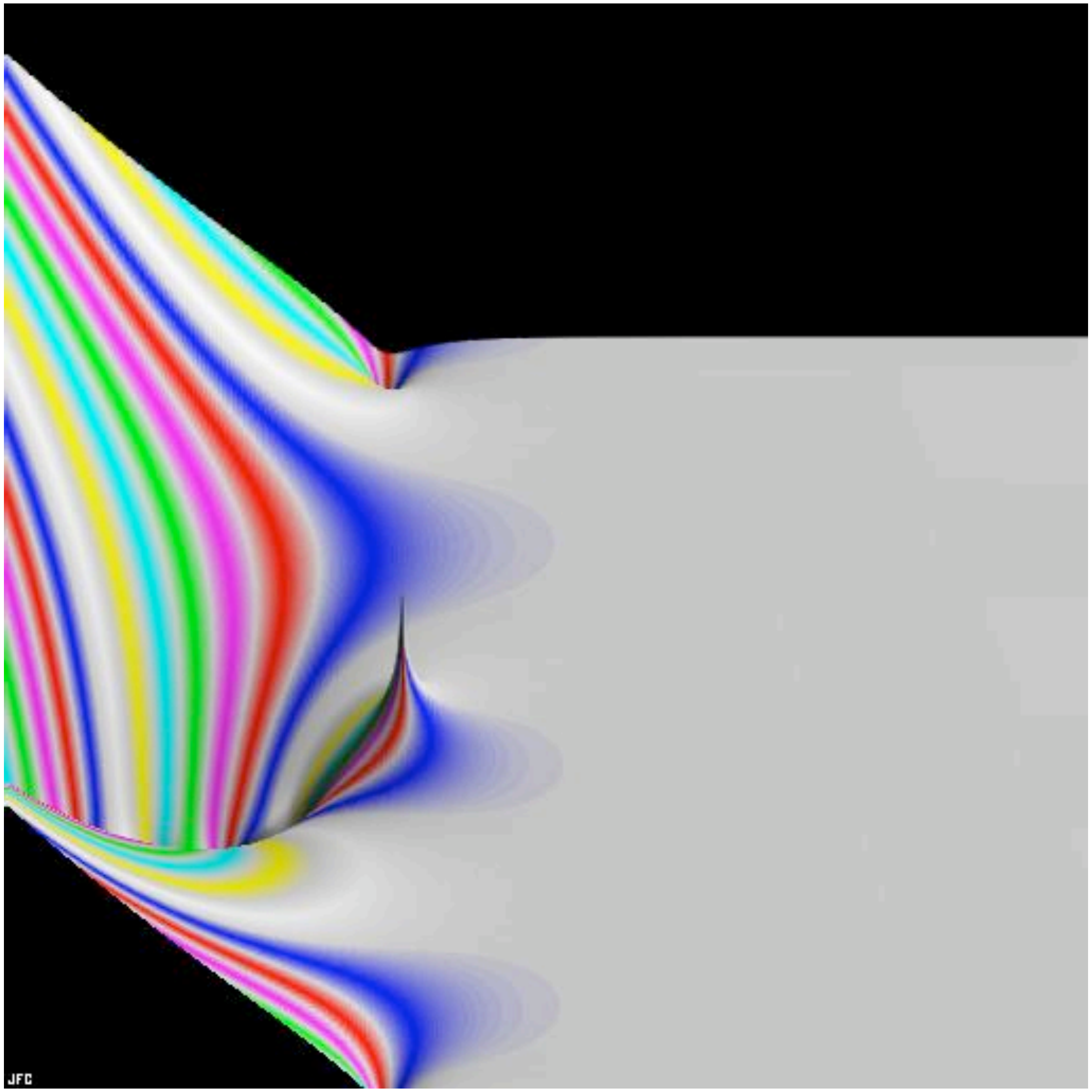
If RH is true then "there is music in the primes"   (M. Berry)

Quick look to the zeta function



The Riemann landscape:

The height is the log of the modulus and the color is the phase of $\varsigma(s)$



Counting non trivial zeros

$\boxed{N_R(E):}$ number of Riemann zeros in the box

$$0 < \mathrm{Re}\, s < 1, \quad 0 < \mathrm{Im}\, s < E$$

It is given by $N_R(E) = \langle N(E) \rangle + N_{fl}(E)$

smooth part -> $\langle N(E) \rangle = \frac{1}{\pi} Arg\, \Gamma(\frac{1}{4} + \frac{i}{2} E) - \frac{E}{2\pi} \log \pi + 1$

fluctuation part -> $N_{fl}(E) = \frac{1}{\pi} Arg\, \varsigma(\frac{1}{2} + i\, E)$

In the limit E >> 1 the smooth part behaves as

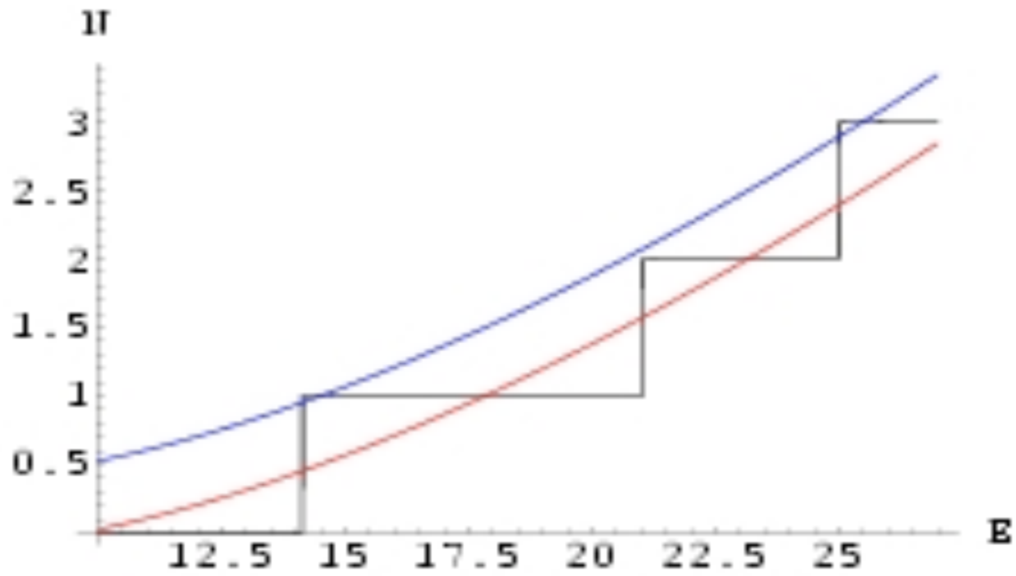
$$\langle N(E) \rangle \approx \frac{E}{2\pi}\left(\log \frac{E}{2\pi} - 1\right) + \frac{7}{8} + \cdots$$

The fluctuation part is $N_{fl}(E) = O(\log E)$

$N_R(E)$: *step function* (*black*)

$\langle N(E) \rangle$: *smooth function* (*red*)

$\langle N(E) \rangle + 1/2$: *smooth function* (*blue*)



$$E_1 = 14{,}134725$$
$$E_2 = 21.022039$$
$$E_3 = 25.010857$$
$$\ldots$$

Duality of the zeta function

Functional relation: $$\varsigma(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \pi^{s-1/2} \varsigma(1-s)$$

Maps the critical line into itself: $$s = \frac{1}{2} + iE \rightarrow 1 - s = \frac{1}{2} - iE$$

$$\varsigma\left(\frac{1}{2} - iE\right) = \pi^{-iE} \frac{\Gamma\left(\frac{1}{4} + \frac{iE}{2}\right)}{\Gamma\left(\frac{1}{4} - \frac{iE}{2}\right)} \varsigma\left(\frac{1}{2} + iE\right) = e^{2i\vartheta(E)} \varsigma\left(\frac{1}{2} + iE\right)$$

$\vartheta(E): phase\ of\ \varsigma(1/2 - iE)$

$$\langle N(E) \rangle = \frac{\vartheta(E)}{\pi} + 1$$

A generalization of zeta(s): Dirichlet L-functions

$\chi(n)$: *Dirichlet character*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}, \quad \Re s > 1$$

Example:

$$L(s,\chi_4) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}, \quad \Re s > 1$$

$$L\left(\frac{1}{2} - iE, \chi_4\right) = \pi^{-iE} \frac{\Gamma\left(\frac{3}{4} + \frac{iE}{2}\right)}{\Gamma\left(\frac{3}{4} - \frac{iE}{2}\right)} L\left(\frac{1}{2} + iE, \chi_4\right)$$

There are even and odd L-functions which come with a 1/4 and 3/4 respectively in the duality relation

All Dirichlet-L functions satisfy the Riemann hypothesis

Three hints for an spectral realization of Riemann zeros

- Montgomery-Odlyzko law
- Berry conjecture
- Selberg trace formula

Montgomery-Odlyzko law
(70´s-80´s):

H: random NxN matrix -> random eigenvalues

Eigenvalues satisfy statistical laws that fall into 3 universality classes described by Random Matrix Theory (RMT)

GOE: gaussian real symmetric matrices
GUE: gaussian hermitean matrices
GSE: gaussian symplectic matrices

MO-law: Riemann zeros satisfy the GUE distribution law

Numerical comparison between
RMT and the statistics of Riemann zeros
(Odlyzko)

Two-point correlation function

Fig 1: first 10^5 zeros

Fig 2: 79 millions of zeros around n= 10^{20}

In physical applications (nuclear physics, condensed matter, etc) the GUE describes systems where the time reversal symmetry is broken

$$Time\ reversal: \quad t \rightarrow -t$$

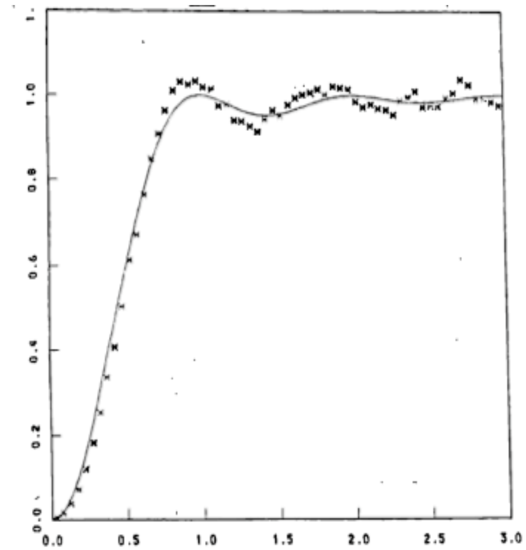


Figure 1.9. Two point correlation function for the zeros $0.5 \pm i\gamma_n$, γ_n real, of the Riemann zeta function; $1 < n < 10^5$. The solid curve is Montgomery's conjecture, Eq. (1.8.9). Reprinted from "On the distribution of spacings between zeros of the zeta function," A.M. Odlyzko, *Mathematics of Computation* pages 273–308 (1987), by permission of The American Mathematical Society.

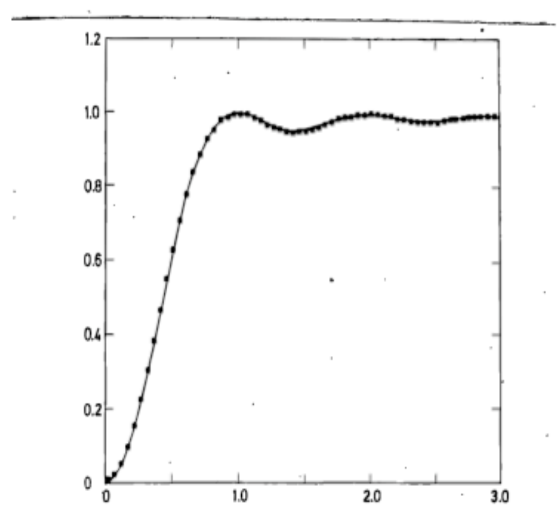


Figure 1.11. The same as Figure 1.9, but for 79 million zeros around $n \approx 10^{20}$. From Odlyzko (1989). Copyright © 1989 American Telephone and Telegraph Company, reprinted with permission.

Conjecture: Riemann Hamiltonian breaks time reversal

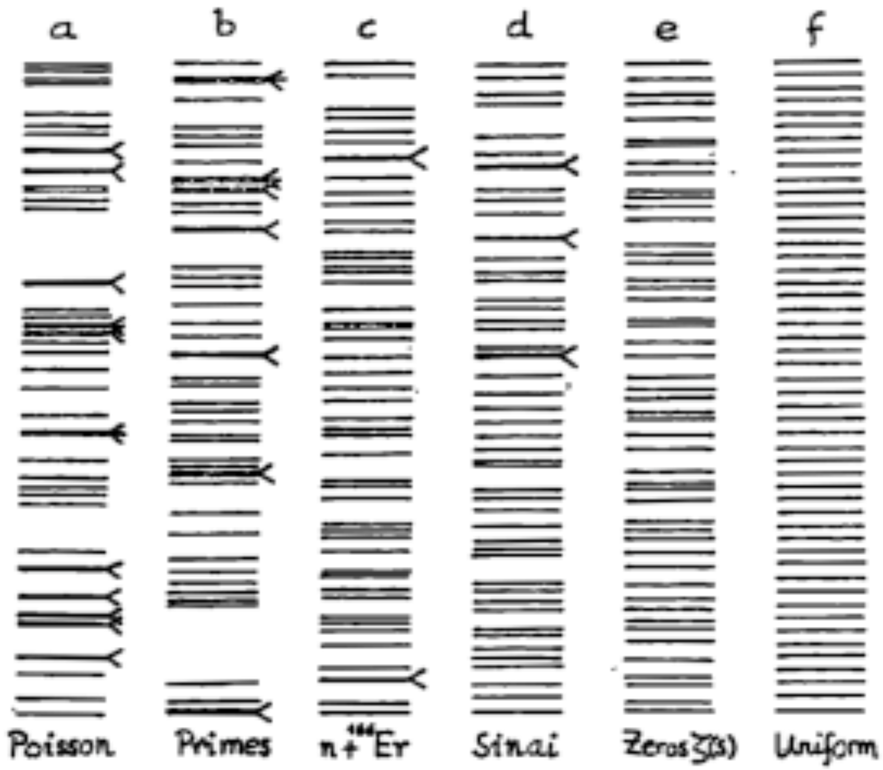


Figure 1.2. Some typical level sequences. From Bohigas and Giannoni (1984). (a) Random levels with no correlations, Poisson series. (b) Sequence of prime numbers. (c) Slow neutron resonance levels of the erbium 166 nucleus. (d) Possible energy levels of a particle free to move inside the area bounded by 1/8 of a square and a circular arc whose center is the mid point of the square; i.e. the area specified by the inequalities, $y \geqslant 0$, $x \geqslant y$, $x \leqslant 1$, and $x^2 + y^2 \geqslant r$. (Sinai's billiard table.) (e) The zeros of the Riemann zeta function on the line $\mathrm{Re}\, z = 1/2$. (f) A sequence of equally spaced levels (Bohigas and Giannoni, 1984).

Berry´s quantum chaos conjecture (80´s-90´s):

Riemann zeros are spectra of a quantum chaotic system

Analogy between the number theory formula:

$$N_{fl}(E) = -\frac{1}{\pi}\sum_{p}\sum_{m=1}^{\infty}\frac{1}{m\,p^{m/2}}\sin\left(m\,E\log p\right)$$

and the fluctuation part of the spectrum of a classical chaotic Hamiltonian

$$N_{fl}(E) = \frac{1}{\pi}\sum_{\gamma}\sum_{m=1}^{\infty}\frac{1}{m\,2\sinh(m\lambda_{\gamma}/2)}\sin\left(m\,E\,T_{\gamma}\right)$$

$$\begin{array}{lcl} \textit{Periodic trayectory}\ (\gamma) & \leftrightarrow & \textit{prime number}\ (p) \\ \textit{Period}\ (T_{\gamma}) & \leftrightarrow & \log p \end{array}$$

$$\begin{array}{ccc} \textit{Prime numbers} & \leftrightarrow & \textit{Riemann zeros} \\ \textit{Time} & \leftrightarrow & \textit{Energies} \\ \textit{Classical} & \leftrightarrow & \textit{Quantum} \end{array}$$

Selberg trace formula (1956)

Riemann surface with negative curvature

Classical problem: compute the length ℓ_p of the geodesics
Quantum problem: compute the spectrum of the Laplacian

$$-\Delta \psi_n(x,y) = E_n \psi_n(x,y), \qquad E_n = \frac{1}{4} + k_n^2$$

There is a "classical-quantum" correspondence given by

$$\sum_n h(k_n) = \frac{\mu(D)}{4\pi} \int_{-\infty}^{\infty} dk\, k\, h(k) \tanh(\pi k) + \sum_{p.p.o.} \ell_p \sum_{n=1}^{\infty} \frac{g(n\ell_p)}{2\sinh(n\ell_p/2)}$$

where h(k) is a test function, g(k) its Fourier transform
The sum is over primitive periodic orbits (p.p.o.)

This reminds the Riemann-Weyl explicit formula

$$\sum_n h(\gamma_n) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} h(k) \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{ik}{2}\right) - 2\sum_{primes} \log p \sum_{n=1}^{\infty} p^{-n/2} g(n \log p) + ctes$$

The Berry-Keating/Connes model

The Berry-Keating/Connes model:

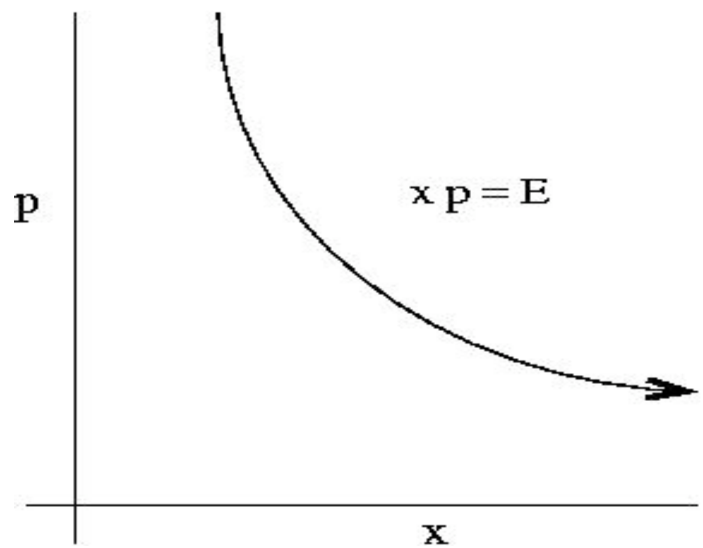
- In 1999 Berry and Keating proposed that the 1D classical Hamiltonian $H = x\,p$, when properly quantized, may contain in the spectrum the Riemann zeros
- This result would imply a proof of the RH
- The Berry-Keating proposal was parallel to Connes adelic approach to the RH.

<u>These approaches are semiclassical</u>

Semiclassical approach to H = xp (Berry-Keating and Connes)

Classically H = x p gives the trayectories

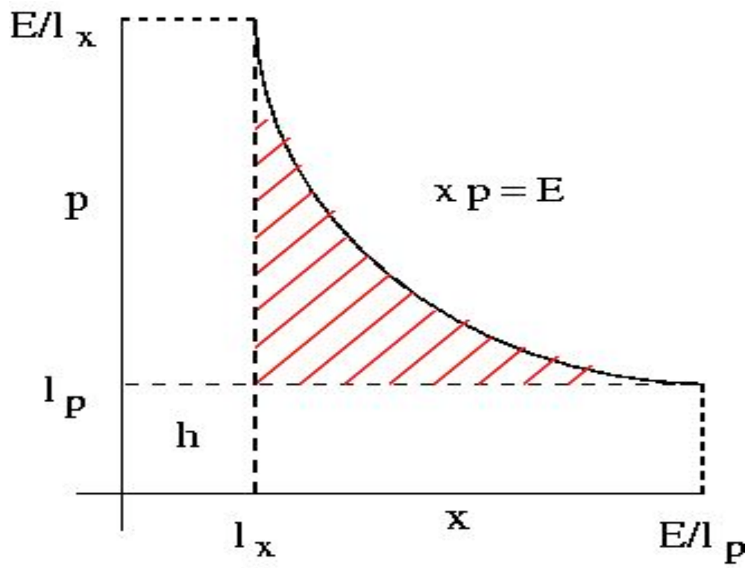
$$x(t) = x_0\, e^{t} \,, p(t) = p_0\, e^{-t} \,, E = x_0\, p_0$$



Time Reversal Symmetry is broken ($x \to x,\ p \to -p$)

Berry-Keating regularization

Planck cell in phase space: $|x| > l_x, \quad |p| > l_p, \quad h = l_x\, l_p = 2\pi \quad (\hbar = 1)$



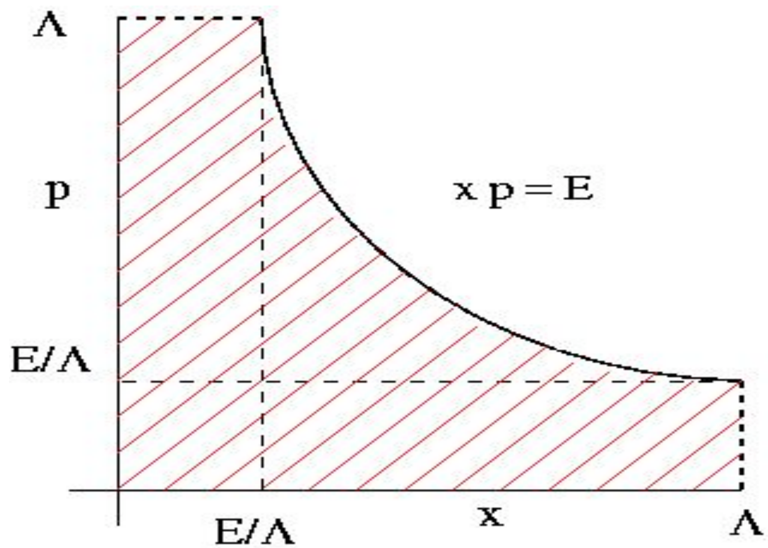
Berry–Keating cutoffs

Number of semiclassical states $N_{sc}(E) \approx \frac{E}{2\pi}\log\frac{E}{2\pi} - \frac{E}{2\pi}$

Agrees with number of zeros asymptotically (smooth part) !!!

Connes regularization

Cutoffs in phase space: $|x| < \Lambda, \quad |p| < \Lambda$



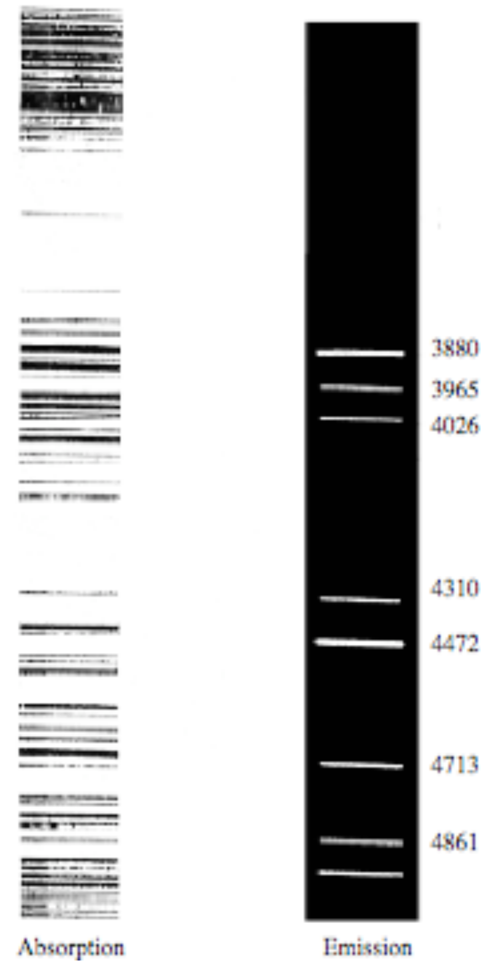
Connes cutoffs

Number of semiclassical states $N_{sc}(E) \approx \frac{E}{2\pi}\log\frac{\Lambda^2}{2\pi} - \frac{E}{2\pi}\log\frac{E}{2\pi} + \frac{E}{2\pi}$

As $\Lambda \longrightarrow \infty$ spectrum = continuum - Riemann zeros

Two spectral scenarios:

Emission (Berry-Keating)
or
Absortion (Connes)



1) Which one is right?

2) Quantum version of these semiclassical models?

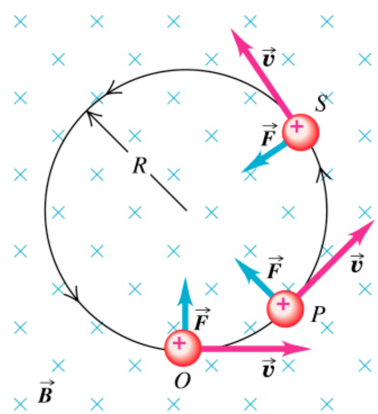
Landau version of the $H = xp$ model

Brief review of the Landau model

Lagrangian of a 2D charge particle in a uniform magnetic field B in the gauge A = B (0,x):

$$L = \frac{\mu}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - \frac{eB}{c}\frac{dy}{dt}x$$

Classically, the particle follows cyclotronic orbits:



(a)

Cyclotronic frequency: $\omega_c = \frac{eB}{\mu c}$

Quantum mechanics: 1D harmonic oscillator

$$H = \hbar\omega_c (a^* a + 1/2), \qquad [a, a^*] = 1, \quad a = \frac{1}{\sqrt{2}}\left(\frac{z}{2} + 2\frac{\partial}{\partial z^*}\right)$$

Landau energy levels

$$E_n = \hbar\omega_c (n + 1/2), \quad n = 0,1,2,\ldots$$

n=0 is the Lowest Landau Level (LLL): huge degeneracy

Semiclassically the n=0 orbits are circles of radius

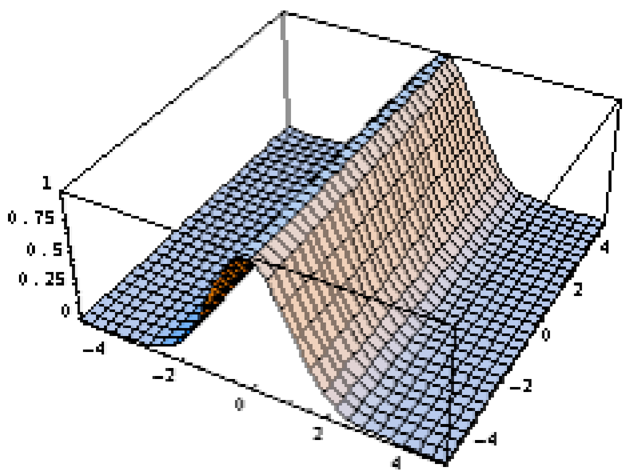
$$\ell = \sqrt{\frac{\hbar c}{eB}} \qquad \text{(magnetic length)}$$

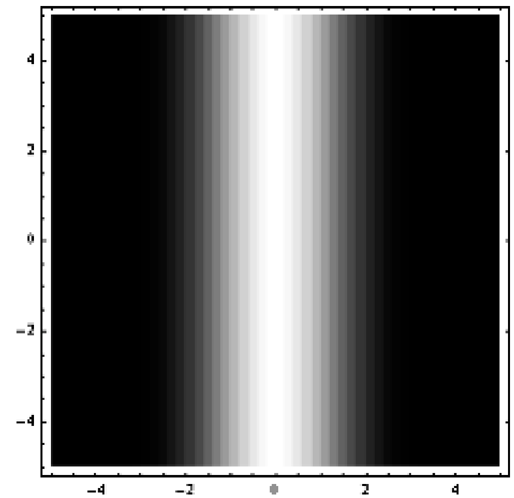
Number of states in the LLL in an area A is

$$N_L = \frac{A}{2\pi\ell^2} = \frac{A \times B}{hc/e} = \frac{\Phi}{\Phi_0} = \frac{Total\ flux}{Quantum\ flux} = N_\Phi$$

Wave functions of the LLL in a box $L_x \times L_y$ (units $\ell = 1$)

$$\psi(x,y) = \frac{1}{\pi^{1/4} L_y^{1/2}} e^{i k_y y} e^{-(x-k_y)^2/2}$$





Boundary conditions $\qquad \psi(x, y+L_y) = \psi(x,y) \Rightarrow k_y = \dfrac{2\pi n}{L_y} \Rightarrow \Delta k_y = \dfrac{2\pi}{L_y}$

Degeneracy $\qquad N_L = \dfrac{L_x}{\Delta x} = \dfrac{L_x}{\Delta k_y} = \dfrac{L_x L_y}{2\pi} = \dfrac{A}{2\pi} = N_\Phi$

Effective Hamiltonian of the LLL

Projection to the LLL is obtained in the limit

$$\omega_c \to \infty \quad \Leftrightarrow \quad \mu \to 0$$

$$L = \frac{\mu}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - \frac{eB}{c}\frac{dy}{dt}x \quad \to \quad L_{LLL} = -\frac{eB}{c}\frac{dy}{dt}x$$

Define $$p = \frac{\hbar y}{\ell^2} \quad \to \quad L_{LLL} = p\frac{dx}{dt}$$

This implies that x and p are the conjugate variables

In the quantum model $$[x, p] = i\hbar$$

$$L_{LLL} = p\frac{dx}{dt} - H_{LLL} \quad \Rightarrow \quad H_{LLL} = 0$$

All the states in the LLL have the same energy

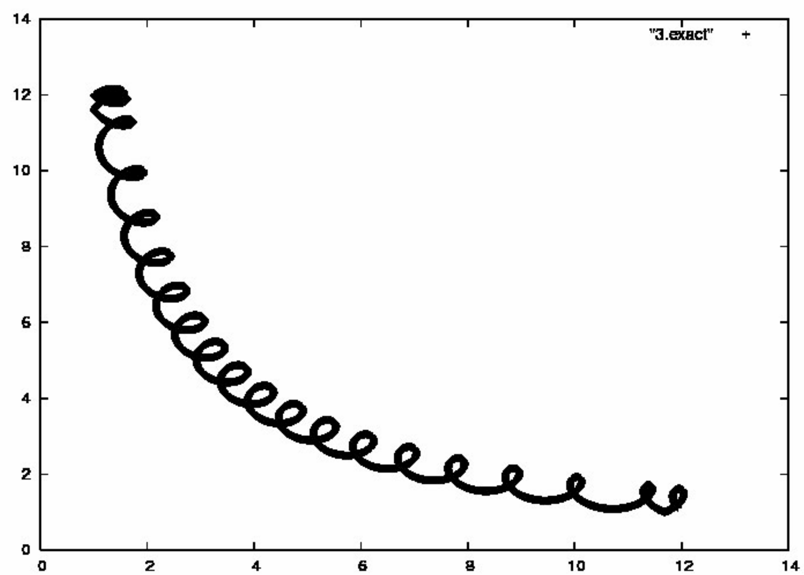
Landau model and Riemann zeros

Add an electrostatic potential xy to the Landau Lagrangian:

$$L = \frac{\mu}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right] - \frac{eB}{c}\frac{dy}{dt}x \quad - e\lambda x y$$

Classical equations of motion have two normal modes: cyclotronic and hyperbolic

$$\omega_c = \frac{eB}{\mu c}\cosh\vartheta, \quad \omega_h = i\frac{eB}{\mu c}\sinh\vartheta \quad \left(\sinh(2\vartheta) = \frac{2\lambda\mu c^2}{eB^2}\right)$$



In the limit $\omega_c >> |\omega_h|$ $\qquad \omega_c \approx \frac{eB}{\mu c}, \quad \omega_h \approx i\frac{\lambda c}{B}$

and only the lowest Landau level is relevant

The effective Lagrangian of the LLL is:

$$\boxed{L_{LLL} = p\,\dot{x} - |\omega_h|\,x\,p, \qquad p = \frac{\hbar\,y}{\ell^2}, \quad \ell^2 = \frac{\hbar c}{eB}}$$

Hence the effective Hamiltonian of the LLL is:

$$\boxed{H_{LLL} = |\omega_h|\,x\,p}$$

Which is the classical Berry-Keating-Connes Hamiltonian!!

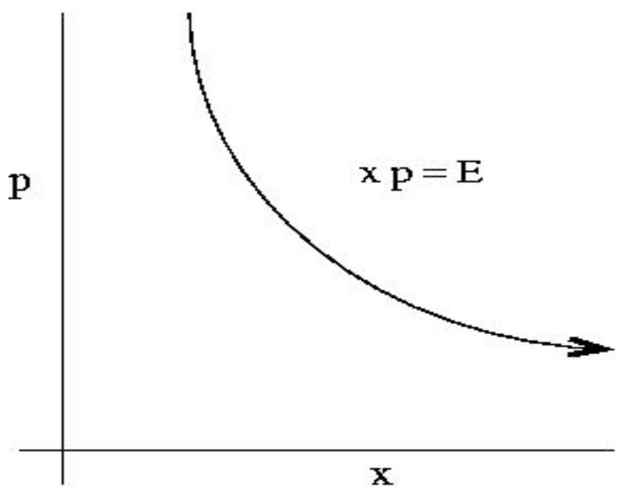
Interpretation of the xp model from the Landau model perspective

Recall the classical trayectories of the xp model

$$x(t) = x_0\, e^{t} \;, p(t) = p_0\, e^{-t} \;, E = x_0\, p_0$$

Since $p = \dfrac{\hbar\, y}{\ell^2}$, they represent the hyperbolic motion of the electron

$$x(t) = x_0\, e^{t} \;, y(t) = y_0\, e^{-t} \;, E = \frac{x_0\, y_0}{\ell^2} \quad (|\hbar\omega_h| = 1)$$



The phase space of the xp model becomes essentially the x-y plane

Semiclassical counting of states

Put the electron in a box LxL $\quad |x| < L, \quad |y| < L$

is analogue to Connes regularization of H= xp

There is a maximum for the classical energy in units of $\hbar\omega_h$

$$|E| < \frac{L^2}{\ell^2}$$

For large L/ℓ the number of semiclassical states with energy less than E is the number of quantum fluxes in the area below the curve

$$x\, y = E\, \ell^2$$

$$N_{sc}(E) = \frac{E}{2\pi}\log\frac{L^2}{2\pi\ell^2} - \frac{E}{2\pi}\left(\log\frac{E}{2\pi} - 1\right)$$

which agrees with Connes counting formula

Quantum derivation of Connes semiclassical result

$$H = \frac{1}{2\mu}\left[p_x^2 + \left(p_y + \frac{\hbar}{\ell^2}x\right)^2\right] + e\lambda x y$$

There is a unitary transformation to $H = H_c + H_h$

$$H_c = \frac{\omega_c}{2}\left(p^2 + q^2\right), \quad H_h = \frac{|\omega_h|}{2}(QP + PQ)$$

In the limit $\omega_c >> |\omega_h|$ the transformation simplifies

$$q = x + p_y, \; p = p_x, \; Q = -p_y, \; P = y + p_x$$

Eigenfunctions of H: $\psi_E^{\pm}(q,Q) = e^{-q^2/2\ell^2} \times \phi_E^{\pm}(Q)$

Eigenfunctions of Hh: $\phi_E^+(Q) = \frac{1}{|Q|^{1/2-iE}}, \quad \phi_E^-(Q) = \frac{\mathrm{sgn}(Q)}{|Q|^{1/2-iE}}$

Quantization of H = xp

Define the normal ordered operator in the half-line

$$H_0 = \frac{1}{2}(x\,p + p\,x) = -i(x\frac{d}{dx} + \frac{1}{2}) \qquad 0 < x < \infty$$

H is a self-adjoint operator: eigenfunctions

$$\phi_E(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{x^{1/2-i\,E}}$$

Normalization appropiate to the continuum $E \in \Re$

$$\langle \phi_E | \phi_{E'} \rangle = \int_0^\infty dx\ \phi_E^*(x)\ \phi_{E'}(x) = \delta(E - E')$$

On the real line H is doubly degenerate with even and odd eigenfunctions under parity

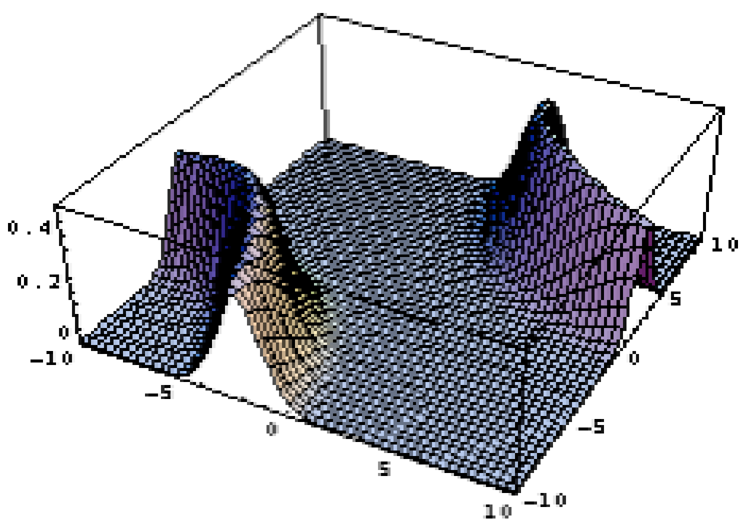
$$\phi_E^+(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{|x|^{1/2-i\,E}}, \qquad \phi_E^-(x) = \frac{1}{\sqrt{2\pi}}\frac{sign(x)}{|x|^{1/2-i\,E}}$$

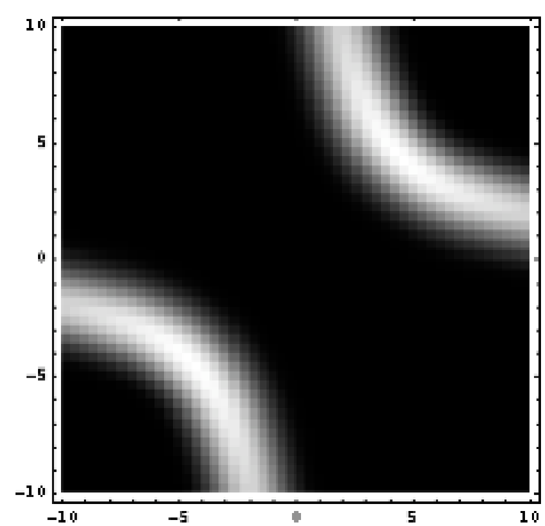
Eigenfunctions of the Landau model

$$\phi_E^{\pm}(x,y) = C \int dQ \, e^{-iQy/\ell^2} \, e^{-(x-Q)^2/2\ell^2} \, \phi_E^{\pm}(Q)$$

$$\phi_E^{+}(x,y) = C_E^{+} \, e^{-x^2/2\ell^2} M\left(\frac{1}{4} + \frac{iE}{2}, \frac{1}{2}, \frac{(x-iy)^2}{2\ell^2}\right)$$

$$\phi_E^{-}(x,y) = C_E^{-} \, (x-iy) \, e^{-x^2/2\ell^2} M\left(\frac{3}{4} + \frac{iE}{2}, \frac{3}{2}, \frac{(x-iy)^2}{2\ell^2}\right)$$





Matching condition on the boundaries (even functions)

$$\phi_E^+(x,L) = e^{iLx/\ell^2}\,\phi_E^+(L,x) \;\rightarrow\; \frac{\Gamma\left(\frac{1}{4}+\frac{iE}{2}\right)}{\Gamma\left(\frac{1}{4}-\frac{iE}{2}\right)}\left(\frac{L^2}{2\ell^2}\right)^{-iE} = 1$$

Taking the log and L >>1

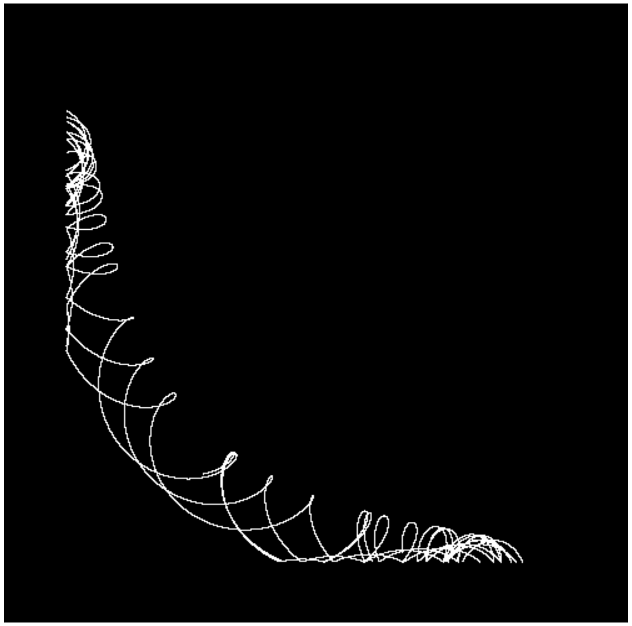
$$N(E) = \frac{E}{2\pi}\log\frac{L^2}{2\pi\ell^2} + 1 - \langle N(E)\rangle$$

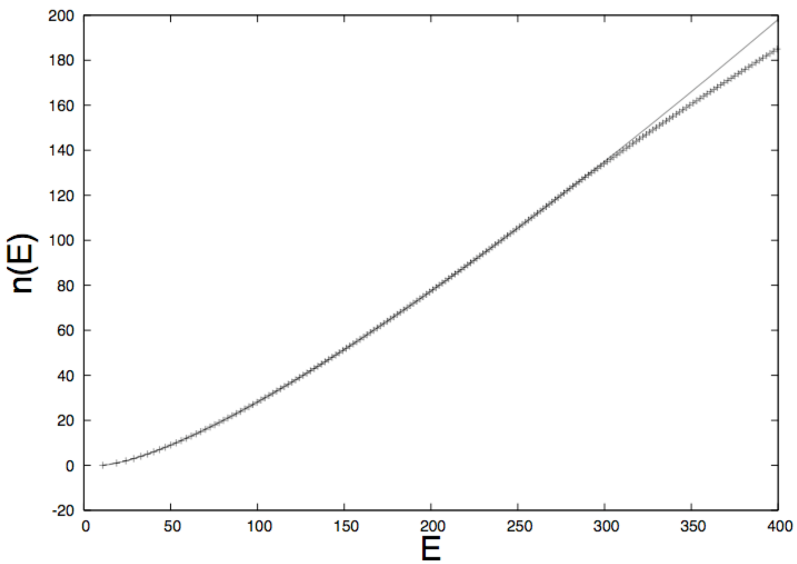
This is the Connes formula

Landau realization of Berry-Keating model
(work in progress with P. Townsend and J. Rodriguez-Laguna)

Restrict the electron to move on the quadrant $x \geq \ell, \qquad y \geq \ell$

Impose periodic boundary conditions $\psi(\ell,t) = \psi(t,\ell), \quad \forall\, t \geq \ell$



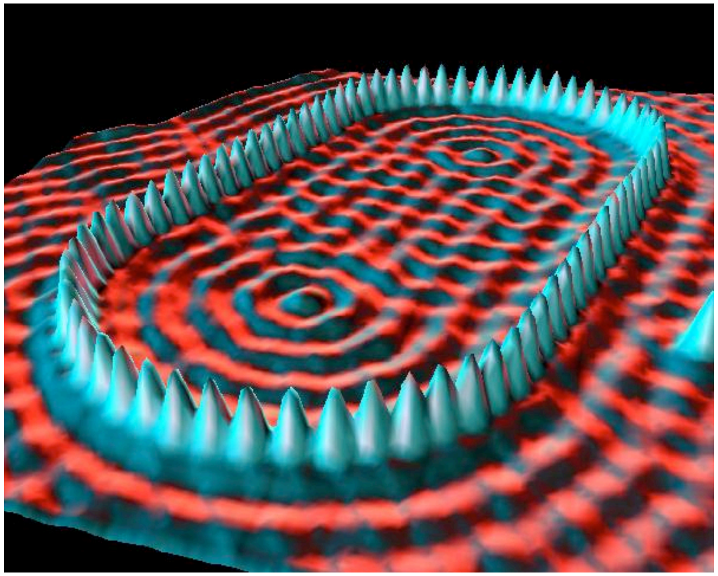


The smooth Riemann zeros appear as eigenstates

Berry has said that if the dynamical system related to the Riemann zeros can be identified, then he is

"absolutely sure that someone will find a clever way to make it in the lab. Then you'll get the Riemann zeros out just by observing its spectrum".

A possibility suggested by the Landau approach for this Lab realization is a quantum corral trapping a single electron:



Number of zeros $\approx 2.5 \times 10^{15} \quad (B = 1 Tesla, \quad Area = 1 m^2)$

Conclusions and suggestions

- We have given a quantum mechanical version of the H = xp model as the effective Hamiltonian of an electron moving in 2D under the action of a uniform magnetic field and a electrostatic potential.
- Connes version of xp is recovered quantum mechanically putting the electron in a box.
- Berry-Keating version of xp seems to be recovered imposing certain periodic boundary conditions.
- The later version seems more promising but further work is needed to clarify this issue
- The Riemann Hypothesis will continue to be an open problem for a while….