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On meromorphic functions defined by a differential system of order 1

Let X denote a complex analytic manifold of dimension n ($n \geq 2$). Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X, $\mathcal{D}_X$ the sheaf of differential operators with holomorphic coefficients and $F_\bullet\mathcal{D}$ its filtration by order. At a point $m \in X$, we will identify the stalk $\mathcal{O}_{X,m}$ (resp. $\mathcal{D}_{X,m}$) with the ring $\mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$ (resp. $\mathcal{D} = \mathcal{O}\langle \partial/\partial x_1, \ldots, \partial x_n \rangle$).

In this lecture, $D \subset X$ denotes a divisor on X, and $h_D \in \mathcal{O}$ will be a reduced equation of D in a local chart.

1 Introduction

Let us recall the result given at the end of the first lecture.

THEOREM 1.1 ([3]) *Let $D \subset X$ be a Koszul-free divisor. Then the inclusion:*

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(\star D)$$

is a quasi-isomorphism if and only if the natural morphism

$$\varphi_D : \widetilde{\mathcal{M}}(\log D) = \mathcal{D}_X \otimes_{\mathcal{V}_0^D(\mathcal{D})} \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(\star D)$$

is an isomorphism.

Here $\mathcal{O}_X(\star D)$ is the $\mathcal{D}_X$-Module of meromorphic functions with poles along D. Locally, this morphism φ_D is:

$$\begin{array}{rcl} \varphi_D : \mathcal{D}/\tilde{I}^{log} & \longrightarrow & \mathcal{O}[1/h_D] \\ P + \tilde{I}^{log} & \longmapsto & P \cdot \dfrac{1}{h_D} \end{array}$$

where $\tilde{I}^{log} \subset \mathcal{D}$ is the left ideal generated by $\mathrm{Ann}_{\mathcal{D}}\, 1/h_D \cap F_1\mathcal{D}$. Moreover, it is bijective if and only if the two conditions are verified:

A$(1/h_D)$**:** The left ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/h_D \subset \mathcal{D}$ of operators annihilating $1/h_D$ is generated by operators of order 1.

B(h_D)**:** The $\mathcal{D}$-module $\mathcal{O}[1/h_D]$ is generated by $1/h_D$.

The aim of this course is to explain as simply as possible what means these conditions. We need to recall some facts about Bernstein polynomials.

2 On the Bernstein polynomial

2.1 Definition and properties

Let $f \in \mathcal{O}$ be a nonzero germ of holomorphic function. We denote by $\mathcal{D}[s]$ the ring $\mathcal{D} \otimes_{\mathbf{C}} \mathbf{C}[s]$ of differential operators with coefficients in $\mathcal{O}[s]$, where s is a new variable.

Theorem 2.1 ([5]) *There exists a functional equation such that:*

$$b(s) f^s = P(s) \cdot f^{s+1} \tag{1}$$

where $P(s) \in \mathcal{D}[s]$ and $b(s) \in \mathbf{C}[s]$ are nonzero.

This identity is verified in $\mathcal{O}[1/f, s] f^s$, where f^s is a formal symbol which twists the natural $\mathcal{D}$-structure of $\mathcal{O}[1/f, s]$. For example:

$$\frac{\partial}{\partial x_i} \cdot a f^s = a'_{x_i} f^s + s \frac{f'_{x_i}}{f} a f^s$$

where $a \in \mathcal{O}[1/f, s]$ and we denote by f^{s+k} the element $f^k \cdot f^s$, $k \in \mathbf{Z}$.

It is easy to see that the set of polynomials verifying such an identity (1) is an ideal. As $\mathbf{C}[s]$ is a principal ring, this ideal is principal. So, we have:

Definition 2.2 *The* Bernstein polynomial *of a nonzero germ $f \in \mathcal{O}$ - denoted by $b(f^s, s)$ - is the monic generator of the ideal of polynomials $b(s)$ which verifies (1).*

Remark 2.3 If $f(0) \neq 0$ then $b(f^s, s) = 1$ since $f^s = f^{-1} \cdot f^{s+1}$. On the other hand, $(s+1)$ is a factor of $b(f^s, s)$ when $f(0) = 0$. Indeed, if we fix "s=-1" in (1), we get: $b(-1)/f = P(-1) \cdot 1$. In particular, $b(-1) \in f\mathcal{O}$, and then $b(-1)$ is zero (since $f(0) = 0$).

So without loss of generality, we will always assume that $f(0) = 0$. Let us give an elementary example.

Example 2.4 Let f be the first coordinate x_1. From the following identity: $(s+1) x_1^s = (\partial/\partial x_1) \cdot x_1^{s+1}$, $(s+1)$ is a multiple of $b(f^s, s)$. With the help of the previous remark, we can also conclude that $b(f^s, s) = (s+1)$.

Finally, let us recall that M. Kashiwara proved that the roots of $b(f^s, s)$ are strictly negative rational numbers ([5]). More precisely, M. Saito proved the following result.

Theorem 2.5 ([10]) *Let $f \in \mathcal{O}$ be a nonzero germ and let $B_f \subset \mathbf{Q}^-$ denote the set of the roots of $b(f^s, s)$. Then $B_f \subset]-n, 0[$.*

2.2 Interest

Let us give three reasons to be interested in this notion.

2.2.1 Meromorphic continuation

At first, this polynomial was introduced by I. N. Bernstein in order to have a meromorphic continuation of special distributions. For example, given a (n, n)-differential form $\varphi \in C_c^\infty(\Omega)$ with compact support in a neighborood Ω of 0. Let us consider the function: $G_\varphi(\lambda) = \int_\Omega |f|^{2\lambda}\varphi$, $\lambda \in \mathbf{C}$. It is easy to check that it is holomorphic on $\mathrm{Re}(\lambda) > 0$, and that the existence of the Bernstein polynomial $b(s)$ of f allows to get a meromorphic continuation of $G_\varphi(\lambda)$ on $\mathbf{C}$.

Indeed, from the identities: $b(\lambda)f^\lambda = P(\lambda)\cdot f^{\lambda+1}$ and $\overline{b}(\lambda)\overline{f}^\lambda = \overline{P}(\lambda)\cdot\overline{f}^{\lambda+1}$, we get: $b(\lambda)\overline{b}(\lambda)|f|^{2\lambda} = \overline{P}P(\lambda)\cdot|f|^{2\lambda+2}$. Hence, if λ is not a root of $b(s)\overline{b}(s)$, we have:

$$G_\varphi(\lambda) = \frac{1}{b(\lambda)\overline{b}(\lambda)}\underbrace{\int_\Omega |f|^{2(\lambda+1)}(\overline{P}P)^*\varphi}_{G_{(\overline{P}P)^*\varphi}(\lambda+1)}$$

where $G_{(\overline{P}P)^*\varphi}(\lambda+1)$ is holomorphic on $\mathrm{Re}(\lambda) > -1$. By iterating this process, we obtain a meromorphic continuation of $G_\varphi(\lambda)$ on $\mathbf{C}$.

2.2.2 Monodromy of $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$

The more important result about Bernstein polynomial is the link of its roots with the monodromy of the Milnor fibration associated with f. This was first discovered by B. Malgrange for isolated singularities, and generalised by M. Kashiwara in general case ([7], [6]). More precisely, we have the following result.

Theorem 2.6 *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a nonzero germ of holomorphic function. Let $E_f \subset \mathbf{C}$ be the set of the eigenvalues α of the monodromy acting on the cohomology of the fibers of the Milnor fibrations of f around the singular points of $f^{-1}(0)$ close enough to 0.*

Then the application $\lambda \mapsto \exp(-2i\pi\lambda)$ induces a sujective map from B_f to E_f.

2.2.3 Finiteness of $\mathcal{O}[1/f]$

Let us remark that $\mathcal{O}[1/f]$ is not a $\mathcal{O}$-module of finite type (since $f(0) = 0$). However, we have the following result.

Proposition 2.7 *Let $f \in \mathcal{O}$ be a nonzero germ. Then $\mathcal{O}[1/f]$ is a $\mathcal{D}$-module of finite type.*

Proof. If $-\ell$ is the smallest integral root of $b(f^s, s)$, then we have:

$$b(-\ell-k)\frac{1}{f^{\ell+k}} \in \mathcal{D} \cdot \frac{1}{f^{\ell+k-1}}$$

where $b(-\ell-k) \neq 0$ for all $k \in \mathbf{N}^*$, using an identity (1) which realises the Bernstein polynomal of f. In particular $1/f^{\ell+k}$ belongs to $\mathcal{D} \cdot 1/f^{\ell}$, $k \in \mathbf{N}^*$, *i.e.* $\mathcal{D} \cdot 1/f^{\ell} = \mathcal{O}[1/f]$. □

This result is very useful in effective algebraic geometry, when one needs a resolution of $\mathcal{O}[1/f]$. Indeed, the first step is also the computation of the Bernstein polynomial of f in order to determinate its smallest integral root (see [8], [9]). Let us mention that the reverse result is true.

PROPOSITION 2.8 ([1], [5]) *Let $f \in \mathcal{O}$ be a nonzero germ such that $f(0) = 0$ and $\ell \in \mathbf{N}^*$. The following conditions are equivalent :*

1. *The smallest integral root of $b(f^s, s)$ is strictly greater than $-\ell - 1$.*
2. *The $\mathcal{D}$-module $\mathcal{O}[1/f]$ is generated by $1/f^{\ell}$.*

3 LCT(D) and $\mathcal{D}$-modules

From the previous result, the condition $\mathbf{B}(h_D)$ introduced in the introduction just means: -1 *is the only integral root of* $b(h_D^s, s)$ (since its roots are negative.) This fact confirms that the $\mathcal{D}$-Module viewpoint is pertinent in order to get a best understanding of the condition $\mathbf{LCT}(D)$.

Moreover, one can prove that the conditions $\mathbf{A}(1/h_D)$ and $\mathbf{B}(h_D)$ are linked.

PROPOSITION 3.1 *[12] Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. Assume that the ideal $\operatorname{Ann}_{\mathcal{D}} 1/h$ is generated by operators of order one. Then the condition $\mathbf{B}(h)$ is verified.*

EXAMPLE 3.2 If $h = x_1^2 + \cdots + x_4^2$, then $b(h^s, s) = (s+1)(s+2)$ and one can check that $\operatorname{Ann}_{\mathcal{D}} 1/h$ is generated by the operators $x_i(\partial/\partial x_j) - x_j(\partial/\partial x_i)$, $1 \leq i < j \leq 4$, $x_1(\partial/\partial x_1) + \cdots + x_4(\partial/\partial x_4) + 2$ and $(\partial/\partial x_1)^2 + \cdots + (\partial/\partial x_4)^2$.

Thus the condition $\mathbf{A}(1/h_D)$ is a local analogue of $\mathbf{LCT}(D)$ for Koszul-free divisors. Is it true for any germ h ? This is true for the weighted-homogeneous hypersurfaces with an isolated singularity (see [11], [4] and below). Moreover, $\mathbf{A}(1/h_D)$ is true for generic arrangements, and this agrees with Terao's conjecture (see the first lecture and [12]). The general problem is still open. Meanwhile, this gives a hope for a best understanding of $\mathbf{LCT}(D)$, since it is easier to work with condition $\mathbf{A}(1/h)$ than with a morphism of complexes.

4 The condition $\mathbf{A}(1/h)$

Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. We recall here the known facts about the meaning of the condition $\mathbf{A}(1/h)$ (see [12]).

First, we have the following easy criterion.

Lemma 4.1 *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. Assume that the following conditions are verified:*

$\mathbf{H}(h)$**:** *h belongs to the ideal of its partial derivatives.*

$\mathbf{B}(h)$**:** *-1 is the smallest integral root of $b(h^s, s)$.*

$\mathbf{A}(h)$**:** *The ideal $\mathrm{Ann}_{\mathcal{D}}\, h^s$ is generated by operators of order 1.*

Then the ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/h$ is generated by operators of order 1.

Proof. Indeed, we have also a decomposition:

$$\mathrm{Ann}_{\mathcal{D}[s]}\, h^s = \mathcal{D}[s](s - v) + \mathcal{D}[s]\mathrm{Ann}_{\mathcal{D}}\, h^s$$

where v is a vector field such that $v(h) = h$. Moreover, under the condition $\mathbf{B}(h)$, the ideal $\mathrm{Ann}_{\mathcal{D}}\, 1/h$ is obtained by fixing $s = -1$ in a system of generators of $\mathrm{Ann}_{\mathcal{D}[s]}\, h^s$ (Proposition 2.8). $\square$

Reciprocally, what does remain true ? We recall that the implication $\mathbf{A}(1/h) \Rightarrow \mathbf{B}(h)$ is always true. On the other hand, does $\mathbf{A}(1/h)$ imply $\mathbf{H}(h)$? This is true for isolated singularities ([11]), Koszul-free germs, and suspensions of unreduced plane curve $z^N + g(x_1, x_2)$ ([12]). So, one can think that it is always true (?)

Finally, the condition $\mathbf{A}(1/h)$ does not imply $\mathbf{A}(h)$ in general. Indeed, Calderón's example $h = x_1x_2(x_1 + x_2)(x_1 + x_2x_3)$ verifies $\mathbf{LCT}(D)$, $\mathbf{A}(1/h)$, $\mathbf{B}(h)$, $\mathbf{H}(h)$ and not $\mathbf{A}(h)$. So, in the identity:

$$\mathrm{Ann}_{\mathcal{D}}\, \frac{1}{h} = \mathcal{D}(x_1\partial_1 + x_2\partial_2 + 4) + \mathrm{Ann}_{\mathcal{D}}\, h^s$$

we remark that something is happening but we can not understood it with these objects.

Meanwhile, condition $\mathbf{A}(h)$ is not unrealistic. Indeed, we have the following characterization of $\mathbf{A}(1/h)$ for Koszul-free germs.

Theorem 4.2 ([12]) *Let $h \in \mathcal{O}$ be a Koszul-free germ. Then $\mathrm{Ann}_{\mathcal{D}}\, 1/h$ is generated by operators of order one if and only if the conditions $\mathbf{H}(h)$, $\mathbf{B}(h)$ and $\mathbf{A}(h)$ are verified.*

Moreover, with the criterion above, we get:

If h verifies $\mathbf{H}(h)$ and $\mathbf{A}(h)$ then $\mathbf{A}(1/h)$ is equivalent to $\mathbf{B}(h)$.

For example, if h defines an isolated singularity, one can prove that:

$$\operatorname{Ann}_{\mathcal{D}} h^s = \sum_{1\le i<j\le n} \mathcal{D}(h'_{x_j}\partial_i - h'_{x_i}\partial_j)$$

(see [13]). Moreover, if h is weighted-homogeneous, the Bernstein polynomial $b(h^s, s)$ is given by a closed formula (see [13]). In particular, we find exactly the technical condition for $\mathbf{LCT}(D)$ given in the first lecture (Theorem 2.1). For example, if $h = x_1^2 + \cdots + x_n^2$ then we have $b(h^s, s) = (s+1)(s+n/2)$ whereas $\mathbf{LCT}(D)$ is true if and only if n is odd or $n = 2$.

5 The condition $\mathbf{A}(h)$

In this last part, we explain why the condition $\mathbf{A}(h)$ may be considered almost as a geometric condition.

Of course, the condition $\mathbf{A}(h)$ may be considered in terms of the $\mathcal{D}$-module $\mathcal{D}h^s$ (since $\mathcal{D}/\operatorname{Ann}_{\mathcal{D}} h^s \cong \mathcal{D}h^s$). To this end, let us recall some classical notions in D-Module theory.

• A nonzero differential operator $P \in \mathcal{D}$ may be written in a unique way as a finite sum: $\sum_{\alpha=(\alpha_1,\dots,\alpha_n)} p_\alpha (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_n)^{\alpha_n}$, with $p_\alpha \in \mathcal{O}$. The degree of P is also the integer $\deg(P) = \max\{|\alpha| \mid p_\alpha \neq 0\}$, and its *principal symbol* is the homogeneous polynomial $\sigma(P) = \sum_{|\alpha|=\deg(P)} p_\alpha \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n} \in \operatorname{gr}^F \mathcal{D} \cong \mathcal{O}[\xi_1, \dots, \xi_n]$.

For example, if $P = 3x_1(\partial/\partial x_1)^2 - (\partial/\partial x_1)(\partial/\partial x_2) + x_1(\partial/\partial x_2) + 1$, then $\deg(P) = 2$ and $\sigma(P) = 3x_1\xi_1^2 - \xi_1\xi_2$.

• Given a nonzero coherent left ideal $I \subset \mathcal{D}$, we denote by $\operatorname{gr} I \subset \mathcal{O}[\xi]$ the homogeneous ideal generated by the polynomials $\sigma(P)$, $P \in I$. The *characteristic variety* $\operatorname{char}_{\mathcal{D}} \mathcal{D}/I$ of $\mathcal{D}/I$ is the zero set of $\operatorname{gr} I$ in $T^*\mathbf{C}^n$.

For example, if $I = \mathcal{D}((\partial/\partial x_1), \dots, (\partial/\partial x_n))$, then $\operatorname{gr} I = (\xi_1, \dots, \xi_n)\mathcal{O}[\xi]$ and $\operatorname{char}_{\mathcal{D}} \mathcal{D}/I = \mathbf{C}^n \times \{0\} \subset T^*\mathbf{C}^n$.

For the coherent $\mathcal{D}$-module $\mathcal{D}/\operatorname{Ann}_{\mathcal{D}} h^s \cong \mathcal{D}h^s$, we have the following result, due to M. Kashiwara.

THEOREM 5.1 ([5]) *Let $h \in \mathcal{O}$ be a nonzero germ such that $h(0) = 0$. The characteristic variety of $\mathcal{D}h^s$ coincides with the relative conormal space:*

$$W_h = \overline{\{(x, \lambda \mathrm{d}h(x)) \mid \lambda \in \mathbf{C}\}}$$

Now it easy to check that the following condition implies $\mathbf{A}(h)$:

$\mathbf{W}(h)$ **:** The relative conormal space W_h is defined by linear equations in ξ.

This is the reason why $\mathbf{A}(h)$ may be considered *almost* as a geometric condition on h.

Example 5.2 The condition $\mathbf{W}(h)$ is true when h defines an isolated singularity, and when h locally weighted-homogeneous free germs ([2]).

Remark 5.3 It is easy to check that the condition $\mathbf{W}(h)$ is equivalent to the following one: the kernel of the morphism of graded $\mathcal{O}$-algebra

$$\begin{array}{rcl} \mathcal{O}[X_1,\ldots,X_n] & \longrightarrow & \mathcal{R}(\mathcal{J}_h) \\ X_i & \longmapsto & th'_{x_i} \end{array}$$

is generated by homogeneous elements of degree 1. Here $\mathcal{J}_h$ denotes the jacobian ideal $(h'_{x_1},\ldots,h'_{x_n})\mathcal{O}$ and $\mathcal{R}(\mathcal{J}_h)$ is the Rees alebra $\bigoplus_{d\geq 0}\mathcal{J}_h^d t^d$. Following a terminology due to W. Vasconcelos, one says that $\mathcal{J}_h$ is of *linear type* (see [2] for more details.)

Remark 5.4 We do not have an example of a germ h verifying $\mathbf{A}(h)$ and not $\mathbf{W}(h)$. Are these conditions equivalent ?

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