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Logarithmic comparison theorem. An introduction

Let X denote a complex analytic manifold of dimension n, ($n \geq 2$). Let $\mathcal{O}_X$ denote the sheaf of holomorphic functions on X. In a point $m \in X$, we will identify the stalk $\mathcal{O}_{X,m}$ with the ring $\mathcal{O} = \mathbf{C}\{x_1, \ldots, x_n\}$.

In this lecture, $D \subset X$ denote a divisor on X, and $h_D \in \mathcal{O}$ will be a reduced equation of D in a local chart.

1 Logarithmic de Rham complex associated with D

1.1 Definition

First, let us consider the classical de Rham complex on X:

$$0 \to \Omega_X^0 \xrightarrow{d_0} \Omega_X^1 \xrightarrow{d_1} \cdots \longrightarrow \Omega_X^{n-1} \xrightarrow{d_{n-1}} \Omega_X^n \to 0$$

denoted by $\Omega_X^\bullet$. In local coordinates, an holomorphic p-form $w \in \Omega_X^p$ may be written as a sum $\sum_{1 \leq \alpha_1 < \cdots < \alpha_p \leq n} a_\alpha dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_p}$, $a_\alpha \in \mathcal{O}$. The differential d_p is defined by:

$$d_p(w) = \sum_{i=1}^{n} \sum_{1 \leq \alpha_1 < \cdots < \alpha_p \leq n} (\partial a_\alpha / \partial x_i) dx_i \wedge dx_{\alpha_1} \wedge \cdots \wedge dx_{\alpha_p} \ .$$

If we consider the divisor $D \subset X$, we have also the meromorphic de Rham complex $\Omega_X^\bullet(\star D)$. The definition is the same as above, but the a_α are meromorphic functions with poles along D. We denote by $\mathcal{O}_X(\star D)$ the sheaf of meromorphic functions with poles along D. In particular, $\Omega_X^0(\star D) = \mathcal{O}_X(\star D)$.

There is another de Rham complex associated with D.

Definition 1.1 *A meromorphic p-form $w \in \Omega^p(\star D)$ is* logarithmic *if w, dw have at most a simple pole along D.*

In local coodinates, this may be written: $h_D w$, $h_D dw \in \Omega_X^\bullet$. For all p, $0 \leq p \leq n$, we denote $\Omega_X^p(\log D) \subset \Omega_X^p(\star D)$ the sub $\mathcal{O}_X$-module of logarithmic forms. It is easy to verify that $\Omega_X^\bullet(\log D)$ is again a complex. It is the so-called *logarithmic de Rham complex* associated with the divisor D. This general definition is due to K. Saito ([10]).

Remark 1.2 The condition "w is a logarithmic form" is not very explicit. For example, let us assume that D is the normal crossing divisor defined by $h_D = x_1x_2$ in $X = \mathbf{C}^n$. Then $w = dx_2/x_1$ has of course a simple pole along D, but w is not logarithmic since $dw = (-1/x_1^2)dx_1 \wedge dx_2$.

Let us recall that the logarithmic de Rham complex was first introduced by P. Deligne in the case of normal crossing divisors ([7]). The $\mathcal{O}_X$-modules $\Omega^1_X(\log D)$ are also quite explicit. In our example above, $\Omega^1_X(\log D)$ is the free $\mathcal{O}_X$-module: $\mathcal{O}_X dx_1/x_1 \oplus \mathcal{O}_X dx_2/x_2 \oplus \mathcal{O}dx_3 \oplus \cdots \oplus \mathcal{O}dx_n$ and for all $1 \leq q \leq n$, $\Omega^p_X(\log D) = \bigwedge_{i=1}^p \Omega^1_X(\log D)$. In general, it is not easy at all to explicit $\Omega^p_X(\log D)$...

1.2 Some properties of $\Omega^\bullet_X(\log D)$

- $\Omega^0_X(\log D) = \mathcal{O}_X$; $\Omega^n_X(\log D) = (1/h_D)\Omega^n_X$.

- $dh_D/h_D \in \Omega^1_X(\log D)$. So the inclusions $\Omega^p_X \subset \Omega^p_X(\log D)$, $p \geq 1$, are strict.

- For all p, the $\mathcal{O}_X$-modules $\Omega^p_X(\log D)$ are coherent (this is an easy exercice, using that $w \in \Omega^p_X(\log D) \Leftrightarrow w$, $dh_D \wedge w$ have almost a simple pole.)

- If $D = D_1 \cup D_2$, then $h_{D_2}\Omega^p_X(\log D) \subset \Omega^p_X(\log D_1) \subset \Omega^p_X(\log D)$.

Definition 1.3 *A holomorphic vector field v is* logarithmic *along D if, for any point $m \in D$, the derivation $v(h_D)$ belongs to $h_D\mathcal{O}_{X,m}$.*

We denote $\mathrm{Der}_X(\log D) \subset \mathrm{Der}_X(\mathcal{O}_X)$ the $\mathcal{O}_X$-Module (coherent) of logarithmic vector fields.

Proposition 1.4 *The inner multiplication of vector fields and differential forms induces a complete pairing of $\mathcal{O}_X$-Modules:*

$$\mathrm{Der}_X(\log D) \times \Omega^1_X(\log D) \longrightarrow \mathcal{O}_X \ .$$

In particular, $\mathrm{Der}_X(\log D)$ and $\Omega^1_X(\log D)$ are $\mathcal{O}_X$-dual.

1.3 What is the Logarithmic Comparison Theorem ?

Let j denote the natural inclusion $X \backslash D \hookrightarrow X$. From Grothendieck Comparison Theorem ([8]), the de Rham morphism:

$$\Omega^\bullet_X(\star D) \longrightarrow \mathbf{R}j_*\mathbf{C}_{X\backslash D}$$

is a quasi-isomorphism. In other words, the meromorphic de Rham complex calculates the cohomology of $X \backslash D$.

Following F.J. Castro-Jiménez, D. Mond and L. Narváez-Macarro ([5]), one says that the divisor D *satisfies the Logarithmic Comparison Theorem* if:

LCT(D) : The inclusion $\Omega^\bullet_X(\log D) \hookrightarrow \Omega^\bullet_X(\star D)$ is a quasi-isomorphism.

For example, P. Deligne proved that **LCT(D)** is verified for normal crossing divisors ([7]) (and this is also an easy exercice to find the cohomology of $H^\bullet(\mathbf{C}^n \backslash D, 0)$ using the complex $\Omega^\bullet_X(\log D)$.)

Remark 1.5 In fact. P. Deligne considered a filtred morphism:

$$(\Omega^\bullet_X(\log D), F) \hookrightarrow (\Omega^\bullet_X(\star D), P)$$

where $P^k(\Omega^p_X(\star D)) = \Omega^p_X((p-k+1)D)$ if $p \geq k$ and 0 otherwise, and he proved that the quasi-isomorphism is compatible with filtrations. This fact was crucial when he defined a mixed Hodge structure for a quasi-projective manifold, using the resolution of singularities in order to get normal crossing divisors.

There are few families of divisors for which this condition is understood. Indeed, it is difficult to work directly with the complex $\Omega^\bullet_X(\log D)$, since we do not have in general a description of the logarithmic forms. In the following, we recall the main results about the condition **LCT(D)**.

2 The case of weighted-homogeneous hypersurfaces with an isolated singularity

Let us recall some definitions.

• A divisor D has an *isolated singularity* in $m \in D$ if on a neighborhood of m, the jacobian ideal $J_{h_D} = (h'_{D,x_1}, \dots, h'_{D,x_n})\mathcal{O}_{X,m}$ defines m.

• A polynomial $h \in \mathbf{C}[x] = \mathbf{C}[x_1, \dots, x_n]$ is *weighted-homogeneous* of weight $d \in \mathbf{Q}^+$ for a system $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbf{Q}^{*+})^n$ if h is a (non trivial) $\mathbf{C}$-linear combination of monomials $x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ with $\sum_{i=1}^n \alpha_i \gamma_i = d$. In other words, we have the relation: $\chi(h) = dh$ where χ is the Euler-vector field $\alpha_1 x_1 \partial_1 + \cdots + \alpha_n x_n \partial_n$ associated with α.

For example, $h = x_1^{a_1} + \cdots + x_n^{a_n}$, where $a_1, \dots, a_n \geq 2$, is a weighted-homogeneous polynomial which defines an isolated singularity in 0.

• A divisor D is *weighted-homogeneous* in $m \in D$ if there exists an analytic change of coordinates ϕ such that $h \circ \phi$ is a weighted-homogenous polynomial.

As usual in theory of singularities, the case of weighted-homogeneous polynomials h defining an isolated singularity at the origin gives explicit formulas in terms of weights linked to the jacobian algebra $A_h = \mathbf{C}[x]/(h'_{x_1}, \dots, h'_{x_n})$.

Theorem 2.1 ([9]) *Let $h \in \mathbf{C}[x]$ be a weighted-homogeneous polynomial of degree d for a system $\alpha \in (\mathbf{Q}^{*+})^n$. Assume that h defines an isolated singularity at the origin. Let $D \subset \mathbf{C}^n$ be the hypersurface defined by h.*

The following condition are equivalent:

1. *The logarithmic comparison theorem holds for D.*
2. *The link of 0 in D is a $\mathbf{Q}$-homology sphere.*
3. *There is no weighted-homogeneous element in A_h whose weight belongs to the set $\{k \times d - \sum_{i=1}^{n} \alpha_i \ ; \ 1 \leq k \leq n-2\} \subset \mathbf{Q}$.*

In particular, the logarithmic comparison theorem does not hold in general. For example, if $h = x_1^2 + \cdots + x_n^2$ then we can take $d = 2$, $\alpha_1 = \cdots = \alpha_n = 1$, and $A_h = \mathbf{C}\bar{1}$. Thus $\mathbf{LCT}(D)$ is also verified if and only if $n = 2$ or n is odd.

3 The case of hyperplane arrangements

Let D be a finite union of affine hyperplanes $H \subset X = \mathbf{C}^n$, *i.e.* $H = \{\alpha_H = 0\}$ where $\alpha_H \in \mathbf{C}[x_1, \ldots, x_n]$ are polynomials of degree one. We can associate with D the $\mathbf{C}$-subalgebra of $\Omega_X^\bullet(\star D)$ generated by 1 and by the 1-forms $d\alpha_H/\alpha_H \in \Omega_X^1(\star D)$. Let $R^\bullet(D)$ denote this algebra of differential forms.

It is well known that $R^\bullet(D)$ is isomorphic to the so called Orlik-Solomon algebra. Moreover, we have:

Theorem 3.1 (E. Brieskorn, 1973) *For all $k \geq 0$, $R^k(D) \cong H^k(X \backslash D, \mathbf{C})$.*

On the other hand, we can consider the complex of $\mathbf{C}$-vector spaces: $0 \to R^0(D) \xrightarrow{0} \cdots \xrightarrow{0} R^n(D) \to 0$ as a subcomplex of $\Omega_X^\bullet(\log D)$. Thus, a natural question is: does the logarithmic comparison theorem hold for any hyperplane arrangement ? This was conjectured by H. Terao (1978). This is true for particular families of hyperplane arrangements (like generic arrangements or complex reflexion arrangements) and when $n \leq 4$ ([12]). But in general, the question is still open.

4 The case of free divisors

4.1 Definition and example

Definition 4.1 ([10]) *A divisor $D \subset X$ is* free *if $\mathrm{Der}_X(\log D)$ is locally free.*

From the inclusions: $h_D \mathrm{Der}_X(\mathcal{O}_X)_m \subset \mathrm{Der}_X(\log D)_m \subset \mathrm{Der}_X(\mathcal{O}_X)_m$, the rank of $\mathrm{Der}_X(\log D)$ is also equal to n.

Example 4.2 Free divisors appear in quite distinct contexts.

1 Normal crossing divisors are free. Indeed, in local coordinates such that $h_D = x_1 \cdots x_p$, then:

$$\mathrm{Der}_X(\log D) = \mathcal{O}x_1\partial_1 \oplus \cdots \oplus \mathcal{O}x_p\partial_p \oplus \mathcal{O}\partial_{p+1} \oplus \cdots \oplus \mathcal{O}\partial_n$$

2 Plane curves are free ([10]).

3 Complex reflexion arrangements are free (H. Terao, 1980). For example, the braid arrangement - defined by $\prod_{1\leq i<j\leq n}(x_i - x_j)$ in $\mathbf{C}^n$ - is free.

4 The discriminant of a semi-universal deformation of an isolated hypersurface singularity is a free divisor (K. Saito, 1983). For example, $F(x,z) = x^4 + z_2x^2 + z_3x + z_4$ is a semi-universal deformation of $f = x^4$. Thus $h_D = \mathrm{disc}(F, F'_x)$ defines a free divisor.

Let us remark that the knowledge of the freeness of a divisor does not give a basis of $\mathrm{Der}_X(\log D)$... However, the following result - 'Saito Criterion' ([10]) - allows to test the freeness of a divisor.

Proposition 4.3 *The divisor D is free in m if and only if there exist $v_1,\dots,v_n \in \mathrm{Der}_X(\log D)$ with $v_i = \sum_{j=1}^n a_{i,j}(\partial/\partial x_i)$ such that $\det(a_{i,j}) = h_{D,m}$ up to a unit.*

The family $\{v_1,\dots,v_n\}$ is also a basis of $Der_X(log\, D)$.

For example, it is easy to prove that the braid arrangement is free.

Most of the known results about the condition $\mathbf{LCT}(D)$ were obtained for free divisors. Indeed, the logarithmic de Rham complex $\Omega_X^\bullet(\log D)$ is also explicit. More precisely, with the duality between $\Omega_X^1(\log D)$ and $\mathrm{Der}_X(\log D)$ (Proposition 1.4), $\Omega_X^1(\log D)$ is also a free $\mathcal{O}_X$-Module and: $\Omega_X^q(\log D) = \wedge^q\Omega_X^1(\log D)$ for $1 \leq q \leq n$ ([10]).

4.2 Main results

First, the situation is well understood if $n = 2$.

Theorem 4.4 ([2]) *If $D \subset X = \mathbf{C}^2$ is a plane curve, then the logarithmic comparison theorem holds if and only if D is locally weighted-homogeneous.*

This last condition means: for all $m \in D$, D is weighted-homogeneous at m. More generally, we have:

Theorem 4.5 ([5]) *Let $D \subset X$ be a locally weighted-homogeneous free divisor. Then the logarithmic comparison theorem holds for D.*

Among the free divisors of Example 4.2, the one given in **3** and some of **4** are locally weighted-homogeneous. Under this strong condition, D is locally a product: $(X,D,m) \equiv (\mathbf{C}^{n-1}\times\mathbf{C}, \mathbf{C}^{n-1}\times\{0\},(0,0))$, and an induction on the dimension may be done. The reverse relation is false in general. For example, $h = x_1x_2(x_1+x_2)(x_1+x_2x_3)$ defines a free divisor for which $\mathbf{LCT}(D)$ is true and h is not weighted-homogeneous ([1]). Meanwhile, M. Schulze proved that a weak form of homogeneity is always necessary.

Theorem 4.6 ([11]) *Let $D \subset X$ be a free divisor. If the logarithmic comparison theorem holds for D, then h_D belongs to the ideal of its partial derivatives at any $m \in D$.*

In other words, there exists locally a vector field v such that $v(h_D) = h_D$. One says sometimes that h is *Euler-homogeneous*. Of course, this condition is in general weaker than weighted-homogeneity (consider $e^{x_3}(x_1^5 + x_2^5 + x_1^2x_2^2)$). However, we recall that this is the same notion for a germ which defines an isolated singularity (K. Saito, 1971).

In conclusion, a form of homogeneity appears to be important in relation to the condition $\mathbf{LCT(}D\mathbf{)}$.

5 A differential viewpoint for free divisors

Given a complex analytic manifold X of dimension $n \geq 2$, we denote $\mathcal{D}_X$ the sheaf of linear differential operators with holomorphic coefficients and $F_\bullet\mathcal{D}$ its filtration by order. Locally at a point $m \in X$, we have $\mathcal{D}_{X,m} \cong \mathcal{D} = \mathcal{O}\langle\partial_1, \ldots, \partial_n\rangle$, and we identify $\mathrm{gr}^F\mathcal{D}$ with the polynomial ring $\mathcal{O}[\xi] = \mathcal{O}[\xi_1, \ldots, \xi_n]$.

Here we recall how the condition $\mathbf{LCT(}D\mathbf{)}$ may be understood in terms of $\mathcal{D}_X$-Modules for free divisors $D \subset X$, as it was initiated by F.J. Calderón-Moreno in [1].

The so-called Riemann-Hilbert correspondence of Z. Mebkhout and M. Kashiwara (1984) asserts that there is an equivalence of categories between the category $hr(\mathcal{D}_X)$ of (left) regular holonomic $\mathcal{D}_X$-Modules and the one of perverse sheaves $Perv_X(\mathbf{C})$ on X, using the de Rham functor:

$$\begin{array}{rcl} hr(\mathcal{D}_X) & \longrightarrow & Perv_X(\mathbf{C}) \\ \mathcal{M} & \longmapsto & \mathrm{DR}(\mathcal{M}) = \Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M} \end{array}$$

Roughly speak, a perverse sheaf on X is a special type of complex of sheaves on X which cohomology groups are constructible in $\mathbf{C}$-vector spaces of finite dimension on a stratification of X. For example, $\mathcal{O}_X$ is regular holonomic, and $\mathrm{DR}(\mathcal{O}_X) = \Omega_X^\bullet$ is quasi-isomophic to the constant sheaf $\mathbf{C}_X$ (Poincaré Lemma).

As $\mathcal{O}_X(\star D)$ is regular holonomic, the meromorphic de Rham complex $\mathrm{DR}(\mathcal{O}_X(\star D)) = \Omega_X^\bullet(\star D)$ is a perverse sheaf too. So it is natural to investigate conditions on D in order to get the perversity of $\Omega_X^\bullet(\log D)$. In the case of free divisors, F.J. Calderón-Moreno proved that this is true when for a particular kind of free divisor.

Definition 5.1 *A free divisor $D \subset X$ is* Koszul-free *if there exists locally a base $\{\delta_1, \ldots, \delta_n\}$ of $\mathrm{Der}_X(\log D)$ such that the sequence of principal symbols $(\sigma(\delta_1), \ldots, \sigma(\delta_n))$ is $\mathrm{gr}^F\mathcal{D}$-regular.*

For example, plane curves and locally weighted-homogeneous free divisors are Koszul-free ([10], [3]).

Theorem 5.2 ([1]) *If D is a Koszul-free divisor, then $\Omega_X^\bullet(\log D)$ is a perverse sheaf.*

The analogue of condition $\mathbf{LCT}(D)$ for free divisors was also investigated by the sevillan group around F.J. Castro-Jiménez and L. Narváez-Macarro (see [6], [4]).

THEOREM 5.3 ([4]) *Let $D \subset X$ be a free divisor. Then the inclusion:*

$$\Omega_X^\bullet(\log D) \hookrightarrow \Omega_X^\bullet(\star D)$$

is a quasi-isomorphism if and only if the following conditions are verified:

1. *the complex $\mathcal{D}_X \otimes^L_{\mathcal{V}_0^D(\mathcal{D})} \mathcal{O}_X(D)$ is concentred in degree* 0 *;*

2. *the natural morphism*

$$\varphi_D : \widetilde{\mathcal{M}}(\log D) = \mathcal{D}_X \otimes_{\mathcal{V}_0^D(\mathcal{D})} \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(\star D)$$

   is an isomorphism.

Here $\mathcal{V}_0^D(\mathcal{D}) \subset \mathcal{D}$ is the ring of logarithmic operators (*i.e.* $P \in \mathcal{D}_X$ such that locally $P \cdot (h_D^k) \subset h_D^k \mathcal{O}$ for any integer k), and $\mathcal{O}_X(D)$ is the $\mathcal{V}_0^D(\mathcal{D})$-module of meromorphic functions with almost a simple pole along D. In the particular case of Koszul-free divisors, the condition 1 in the differential characterization above is always verified ([1]).

Finally, let us remark that the surjectivity of the morphism φ_D means locally: $\mathcal{D}1/h_D = \mathcal{O}[1/h_D]$. So this differential viewpoint is very useful in order to have a best understanding of the condition $\mathbf{LCT}(D)$ (since it was not clear at all that this last condition is necessary for $\mathbf{LCT}(D)$ in the case of free divisors).

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